I. INTRODUCTION

WHY STUDY ACCELERATORS?

- PROGRESS IN SCIENCE IS CLOSELY CONNECTED TO THE CAPABILITIES OF INSTRUMENTS

- ACCELERATORS & DETECTORS ARE THE INSTRUMENTS OF PARTICLE PHYSICS.

- HISTORY
  STOCHASTIC COOLING — W, Z
  L+ L- STORAGE RINGS — Z, C, B

- THE FUTURE
  SUPERCONDUCTING MAGNETS — HICCS
  LINEAR COLLIDERS — ??
WHAT IS AN ACCELERATOR?

- CHARGED PARTICLES MOVING IN ELECTROMAGNETIC FIELDS.

- THESE FIELDS CAN BE
  1. STATIC OR TIME DEPENDENT.
  2. EXTERNALLY APPLIED OR BEAM GENERATED.
  3. LINEAR (RESTORING FORCE & DISPLACE FROM EQUILIBRIUM) OR NON-LINEAR

- THE MOTION CAN BE
  CLASSICAL OR QUANTUM MECHANICAL

COMBINATIONS OF ALL THESE OCCUR & CAN BE IMPORTANT.

EXAMPLES
B. THE SSC OR LHC APERTURE - STATIC, EXTERNAL, NON-LINEAR, CLASSICAL
C. B FACTORY LUMINOSITY LIMIT - TIME DEPENDENT, BEAM GENERATED, NON-LINEAR, CLASSICAL.
A. BETATRON MOTION - STATIC, EXTERNAL, LINEAR, CLASSICAL
D. LINEAR COLLIDER INTERACTION REGION-TIME DEP., BEAM GEN., NON-LINEAR, QUANTUM MECH.
- The approach is to first understand the dominant contributions & use this as a basis for perturbations. That dominant motion is - external, linear, classical, static and/or time dependent
II. SINGLE PARTICLE MOTION

LINEAR ACCELERATOR

IDEAL PARTICLE
(i) ON AXIS
(ii) AT THE NOMINAL ENERGY (WHICH INCREASES LINEARLY WITH t)

QUADRUPOLE LENSES

PARTICLES DEVIATE FROM THE IDEAL
- OFF-AXIS, QUADRUPOLES, RETURN TO LATER
- ENERGY - MOTION IS SIMPLE

RF CAVITIES

RF WAVE, PARTICLE, IDEAL ALL TRAVELING AT c.

1. \( \frac{d\vec{r}}{dt} = \frac{\vec{E}}{mc^2} \) \quad \text{G. cos} \, \vec{k}_RF \, \vec{x} \quad \frac{dE}{dt} = \frac{d}{dE} \vec{E} \)

2. \( \frac{dz}{d\phi} = 0 \) \quad z = \text{const} \quad \vec{v} = \vec{c}
1. $\frac{\sigma^2}{\sigma^2}\approx$ FRACT. ENERGY SPREAD $\sim \sigma^2$

2. BUNCH SHAPE IS STABLE.

\[ \frac{\Delta \sigma}{\sigma_{\text{ideal}}} \]

PARTICLE SITS HERE THROUGHOUT ACCEL. CYCLE.
CIRCULAR ACCELERATOR (STORAGE RING)

IDEAL PARTICLE
I) ON-AXIS
II) AT THE NOMINAL ENERGY (WHICH IS CONSTANT)

LOOK AT AN ENERGY DEVIATION

IN RF
\[ \frac{d\gamma}{d\phi} = \frac{eE}{m_0c^2} \cos\left(\frac{\pi}{2} + \omega_{RF} \tau\right) \]
\[ \frac{d\tau}{d\phi} = 0 \]

1 PASSAGE THROUGH RF
\[ \Delta \gamma = -\frac{eGL}{m_0c^2} \omega_{RF} \tau \]
\[ \Delta \tau = 0 \]
BETWEEN RF PASSES
\[ d\phi = Rd\theta \]
\[ d\phi' = R\left(1 + \frac{d\delta}{\delta}\right)d\theta \]
\[ c\phi' = d\phi' - d\phi = \frac{d\delta}{\delta} d\phi \]
\[ \frac{d\phi}{d\phi'} = \frac{1}{2} \frac{d\delta}{\delta} \quad \frac{d\delta}{d\phi} = 0 \]

\[ \downarrow \]
1 PASS BETWEEN RF'

\[ \Delta \phi = T_0 \frac{d\delta}{\delta} \quad \Delta (\Delta \delta) = 0 \]
\[ \text{REVOLUTION PERIOD} \]

THE QUADS COMPLICATE THIS A LITTLE

\[ \Delta \phi = \alpha T_0 \frac{d\delta}{\delta} \quad \Delta (\Delta \delta) = 0 \]
\[ \text{MOMENTUM COMPACTION} \]
\[ \alpha \sim 0.01 \]
AN INDIVIDUAL PARTICLE OSCILLATES ABOUT THE POSITION OF THE IDEAL.

- "SYNCHROTRON OSCILLATION"
- INTERCHANGING ENERGY & TIME DISPLACEMENTS.
- "PHASE STABILITY", E. McMillan, 1945
III. SINGLE PARTICLE MOTION
TRANSVERSE

- THE QUADRUPOLES PROVIDE TRANSVERSE FOCUSING

\[ \vec{B} = -\vec{\nabla} \psi \]

\[ \psi = q \times \gamma \]

GRADIENT

\[ B_x = -q \gamma \]

\[ B_y = -q \gamma \]

\[ \frac{d \vec{p}}{dt} = e \vec{v} \times \vec{B} \]

\[ \chi \int_{\sigma} \vec{v} \cdot \frac{d \chi}{d \sigma} = \chi' \]

IDEAL'S PATH

\[ \vec{v} \cdot d\sigma = c dt \]

\[ \frac{d p_x}{dt} = \frac{8 m_0 c^2 d \chi'}{d \sigma} \]

\[ (\vec{v} \times \vec{B})_z = -c B_3 = c q \gamma \]

\[ \therefore \quad \frac{8 m_0 c^2 d \chi'}{d \sigma} = e c q \gamma \]

\[ \frac{d \chi'}{d \sigma} = \frac{d^2 \chi}{d \sigma^2} = \frac{e q \gamma}{8 m_0 c} \]

13
\[
\frac{d^2x}{dt^2} = \frac{eg}{\delta m_0 c} x = k x
\]

- SIMPLE HARMONIC MOTION IF \( k < 0 \) (FOCUSBING)
- EXPONENTIAL GROWTH IF \( k > 0 \) (DEFOCUSING)

REPEAT FOR \( y \)

\[
\frac{dp_y}{dt} = \delta m_0 c^2 \frac{dy}{dt}
\]

\((\vec{v} \times \vec{B})_y \equiv \vec{c} \beta x = -cg y^\prime\)

so \( y \) Eq. is \( \frac{d^2y}{dt^2} = -\frac{eg}{\delta m_0 c} \gamma = -k y\)

\[
\frac{d^2y}{dt^2} = k y
\]

\[
\frac{d^2x}{dt^2} = k x\]

THESE EQUATIONS SHOW A CENTRAL FEATURE OF STRONG (QUADRUPOLE) FOCUSING.

\( k > 0 \) DEFOCUSING

\( k < 0 \) FOCUSING

\( \Rightarrow \) COMPOUND OPTICAL SYSTEMS
- Principle of "Strong Focusing" is the proof that such compound systems can have stable (bounded, oscillatory) motion in both dimensions.

- Courant & Snyder, 1956.

- This motion is "linear" in sense that equations look like simple harmonic motion locally, but they are "non-linear" in that \( x \) can depend on \( \alpha \)

\[
\frac{d^2x}{dx^2} = \alpha(\alpha) x
\]

- Natural solution to look for

\[
x = A w(\alpha) e^{\pm \psi(\alpha)}
\]

Amplitude

(factor out)
- Ignore bends (storage ring) and acceleration (linear)

\[ \frac{d^2 x}{d\phi^2} = -k(\phi)x \quad x = A \omega(\phi) e^{-\psi(\phi)} \]

\[ \downarrow \]

\[ \frac{d^2 \omega}{d\phi^2} + k(\omega(\phi)) - \frac{1}{\omega^3} = 0 \quad \text{Hill's Eq} \]

\[ \frac{dy}{d\phi} = \frac{1}{\omega^2} \quad \text{(see Courant + Snyder)} \]

- Comments

1. \( \omega \) depends on the "lattice" - the configuration of quadrupoles
2. However, \( \omega \) is not uniquely determined by the lattice - 2nd order O.E. = \( \chi \) 2 constants of integration
3. \( \omega \) has an absolute scale - if \( \omega_1 \) is a solution, \( 2\omega_1 \) is not.
4. Units of \( \omega = \text{my}^2 \)
5. Phase advance of oscillation, \( \frac{dy}{d\phi} \) is proportional to \( 1/\omega^2 \)

Name \( (\omega(\phi))^2 \equiv \beta(\phi) = \text{Beta function} \)
- Storage ring - described by a periodic lattice \( \hat{r}(0) = \hat{r}(0+c) \) \( c = \text{circum} \).

B-function of accelerator is the periodic solution of Hill's Eq.

\[ B(0) = B(0+c) \]

Periodic \( B \implies \text{periodic particle motion} \)

\[ x = A \sqrt{B(0)} e^{\omega t} \psi(0) \]

- Sample lattice - "SSC"

\[ \begin{array}{c}
 F \\
 D
\end{array} \xrightarrow{180^\circ} \begin{array}{c}
 D \\
 F
\end{array} \xrightarrow{5.05m} \begin{array}{c}
 F \\
 D
\end{array} \]

Gradient = 211 T/m

My "SSC" has 500 cells
\[ \beta_0 \text{ FOR 1 CELL} \]
Horiz Traj. SSC FODO Cell

$\sqrt{P_a}$

$\text{PARTICLE}$

TURN 1

$\sqrt{P_a}$

$\text{TURN 2}$

$\sqrt{P_a}$

$\text{TURN 1}$
\[ a = A \sqrt{\beta(0)} e^{\gamma 10} \]

\[ 4 = \int \frac{1}{\beta} \, d\beta \]

1. \( \beta \) IS PERIODIC, TRAJECTORIES AREN'T

2. \( \sqrt{\beta} \) FORMS THE ENVELOPE OF THE TRAJECTORIES.
\[ \chi = A \sqrt{B(0)} \alpha^* \psi(0) \]

\[ \psi(0) = \int \frac{1}{\beta} \, d\phi \]

- FOR ONE COMPLETE TURN

\[ \Delta \psi = \int_{0}^{C} \frac{d\phi}{\beta} \equiv 2\pi I \]

\( Q \) IS THE NUMBER OF OSCILLATIONS/TURN

- \( I \) MEASURES THE SENSITIVITY TO ERRORS

- APPROXIMATE \( I \) AS CONSTANT

\[ \chi = I \sigma \omega (\alpha/\beta) \]

- NEW TRAJECTORY

- QUADRUPOLE ERROR:

\[ R \Delta \alpha \text{ AT } \beta = 0 \]

\[ \chi = (B + \Delta B) \cos \left( \frac{\alpha + \Delta \psi}{\beta} \right) \]

AT THE ERROR \( \Delta \chi = 0 \)

\[ \Delta \chi' = \chi' - \chi = R \Delta \alpha \chi = \left( \frac{B + \Delta B}{\beta} \right) \Delta \alpha \Delta \psi \]

\[ \chi = B \cdot \Delta \psi \Rightarrow \Delta \psi = B \Delta \alpha \beta \]
LARGE $\beta \Rightarrow$ LARGE $\Delta \phi \Rightarrow$ HIGH SENSITIVITY TO ERRORS

INTERACTION REGIONS

TWO ARGUMENTS FOR LOW $\beta$

1. $\alpha = \frac{N^2 \frac{1}{\gamma}}{4\pi \sigma_x \sigma_y}$
   \[ N = \# \text{OF PART/BUNCH} \] \[ \frac{1}{\gamma} = \text{COLLISION FREQ.} \]
   \[ \Rightarrow \text{EFFECTIVE COLLISION AREA} \]

   $\sqrt{\beta}$ MEASURES BEAM ENVELOPE

   $\therefore$ MAKE $\beta$ SMALL TO GET SMALL AREA

2. THE ONCOMING BEAM IS ROUGHLY EQUIVALENT TO AN ERROR

   $\therefore$ MAKE $\beta$ SMALL TO GET LOW SENSITIVITY TO THIS ERROR

AT: COLLISION PT $\beta = \beta^*$

AT: DISTANCE A AWAY $\beta = \beta^* + \frac{q^2}{\beta^*}$

SMALL $\beta^*$ $\Rightarrow$ RAPID BLOW-UP OF BEAM AWAY FROM INTERACTION PT
1. $\beta^* = 0.5 \text{ m}$

2. $20 \text{ m to first quad} \Rightarrow \sqrt{\beta} = 20 \text{ m} / 0.5 \text{ m} = 28 \text{ m}$

3. First quad focuses in only 1 dimension $\Rightarrow$ defocuses in other $\Rightarrow \beta$ grows to almost $10 \text{ km}$! IR quad field quality is critical

4. Normal, repetitive $\beta$'s restored by end of interaction region.
Figure 4.1.1.1-11. Lattice and orbit functions of low-β IR in collision optics.
Ⅲ. EMITTANCES

- IDEAL PARTICLE & MOTION ABOUT THE IDEAL

? HOW BIG IS THAT MOTION?

- PART OF THE ANSWER DEPENDS ON THE ELEMENTS OF THE ACCELERATOR, PART OF THE ANSWER IS MORE BASIC

- EXAMPLE

\[ \chi = A \sqrt{\beta (v)} \cos [4 \psi (v)] \]

\[ \text{DEPENDS ON ACCELERATOR CONFIGURATION} \]

MORE BASIC
- CONSTANT OF MOTION
- \( A^2 \sim \text{ENERGY OF OSCILLATION} \)
- EMITTANCE, \( \varepsilon \), DEFINED AS

\[ \varepsilon = \langle A^2 \rangle \] TYPICAL ENERGY

- \( \sigma_x = \sqrt{\beta \varepsilon} \) RMS BEAM SIZE

ALSO CAN TALK ABOUT A LONGITUDINAL EMITTANCE

\[ \varepsilon_L = \text{RMS LENGTH} \times \text{RMS ENERGY SPREAD} \]
The emittances have different underlying physics. Look at linear collider as an example

\[ \text{Damping Ring} \xrightarrow{\text{Linac}} \text{Collision Pt} \]

- Damping Ring & Linac have different physics. Look at Linac first

Liouville's Theorem

Hamiltonian systems

Phase space volume cannot change in time

Phase space density cannot change in time

\[ q(t_2), p(t_2) \]

\[ q(t_1), p(t_1) \]

Fig. 8-1. Motion of a volume in phase space.
IN THE TRANVERSE

\[ \Delta \rho_x = \Delta \chi \]

\[ \Delta \chi \Delta \rho_x = \varepsilon_I = \text{constant} \]

IN Variant EMITTANCE

\[ = \Delta \chi \varepsilon \Delta \rho_x = \varepsilon \varepsilon \]

EMITTANCE

\[ \varepsilon = \frac{\varepsilon_I}{\varepsilon} \text{ falls like } \frac{1}{\varepsilon} \]

IN THE LONGITUDINAL

\[ \Delta z \Delta \delta = \varepsilon_L = \text{const} \]

LONGITUDINAL EMITTANCE

THE BEAM ENTERS THE LINAC WITH AN \( \varepsilon_I, \varepsilon_L \). LIOUVILLE'S THEOREM SAYS IT LEAVES WITH \( \varepsilon_I, \varepsilon_L \)

- LIOUVILLE'S EMITTANCES CANNOT GROW
- BUT LIOUVILLE'S THEOREM APPLIES TO OCCUPIED PHASE SPACE. COTTON AND ANALOGY =>

\[ \varepsilon_I (\text{end of linac}) \geq \varepsilon_I (\text{beginning}) \]

\[ \varepsilon_L (\text{end}) \leq \varepsilon_L (\text{beginning}) \]
DAMPING RING - NOT A HAMILTONIAN SYSTEM DUE TO SYNCHROTRON RADIATION
- AVERAGE ENERGY LOSS → DAMPING
- FLUCTUATIONS FROM AVERAGE → EMITTANCE GROWTH.

THE EQUILIBRIUM EMITTANCE IS GIVEN BY THE BALANCE OF DAMPING & GROWTH.

SUMMARY

HAMILTONIAN SYSTEMS (LINAC, PROTON ACCELERATORS) -
1. THE MINIMUM EMITTANCE IS DETERMINED BY THE PARTICLE SOURCE
2. ERRORS TEND TO INCREASE THE EMITTANCE & EMITTANCE PRESERVATION IS A MAJOR DESIGN & OPERATIONAL CONSIDERATION.

NON-HAMILTONIAN SYSTEMS (DAMPING RINGS, ELECTRON ACCELERATORS) -
1. THE EMITTANCE IS DETERMINED BY THE PROPERTIES OF SYNCHROTRON RAD.
2. THE ACCELERATOR CAN BE DESIGNED TO HAVE A DESIRED EMITTANCE.