Linearized Vlasov Equation for a Bunched Beam: Removal of Singularity

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Vlasov equation for longitudinal coherent motion:

$$\frac{\partial f}{\partial \theta} + p\frac{\partial f}{\partial q} + \big(-q + F(q,f)\big)\frac{\partial f}{\partial p} = 0 \ ,$$

where

$$\theta = \omega_s t \ , \quad q = \frac{z}{\sigma_z} \ , \quad p = -\frac{E - E_0}{\sigma_E} \ ,$$

and

$$F(q,f) = I \int dq' W(q - q') \int dp f(q', p, \theta) \ .$$

Equilibrium solution:

$$f_0(q,p) = \frac{e^{-p^2/2}}{\sqrt{2\pi}}\rho(q) \ ,$$

where $\rho(q)$ is the solution of the Haïssinski equation.

Transform to angle-action variables (ϕ, J) of the equilibrium Hamiltonian,

$$H_0 = \frac{1}{2}\big(p^2 + q^2\big) + \int_q^\infty F(q', f_0)dq' \ .$$

Put

$$f(\phi, J, \theta) = f_0(J) + f_1(\phi, J, \theta) \ ,$$

linearize about f_0:

$$\frac{\partial f_1}{\partial \theta} + \Omega(J)\frac{\partial f_1}{\partial \phi} - f_0'(J)\frac{\partial H_1}{\partial \phi} = 0 \ ,$$

$$\begin{aligned}\Omega(J) \ &= \ H_0'(J) \ , \\ f_0(J) \ &= \ Ae^{-H_0(J)} \sim e^{-J} \ , \ J \to \infty \ .\end{aligned}$$

Take Laplace transform w.r.t. θ, Fourier transform in ϕ:

$$\begin{aligned}&(s+im\Omega(J))\hat{f}_1(m,J,s)\\&-f_0'(J)\frac{1}{\pi}\int_0^{2\pi}d\phi e^{-im\phi}\frac{\partial}{\partial\phi}F(Q(\phi,J),f_1)\\&=\check{f}_1(m,J,0)\ ,\end{aligned}\tag{8}$$

where

$$\begin{aligned}q\ &=Q(\phi,J)\ ,\quad p=P(\phi,J)\\\hat{f}_1\ (m,J,s)&=\frac{1}{\pi}\int_0^{2\pi}d\phi e^{-im\phi}\int_0^{\infty}d\theta e^{-s\theta}f_1(\phi,J,\theta)\\\check{f}_1\ (m,J,0)&=\frac{1}{\pi}\int_0^{2\pi}e^{-im\phi}d\phi f_1(\phi,J,0).\end{aligned}\tag{9}$$

In (8), integrate by parts on ϕ, put $m=0$ to obtain

$$s\hat{f}_1(0,J,s)=\check{f}_1(0,J,0)\ \rightarrow\ f_1(0,J,\theta)=\text{const.}\ .\tag{10}$$

Since the zero mode is constant, we can put it equal to 0 in the initial value of the perturbation $\check{f}(0,J,0)$ and thereby delete it from the equation for $\hat{f}_1$.

In (8), the force F involves $\int dq'dp'\cdots$ which we replace by $\int d\phi' dJ'\cdots$.

Thus, my heart's desire, an *Integral Equation*:

$$
\begin{aligned}
&(s + im\Omega(J))\hat{f}_1(m, J, s) \\
&+ \sum_{m'} \int dJ' K(m, J, m', J')\hat{f}_1(m', J', s) \\
&= \check{f}(m, J, 0) ,
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
K(m,J,m',J') = \frac{f_0'(J)}{2\pi} \int d\phi e^{-im\phi} \int d\phi' e^{im'\phi'} \\
\cdot Q_1(\phi, J) W(Q(\phi, J) - Q(\phi', J')) .
\end{aligned}
\tag{12}
$$

Put $s = -i\omega$; then the right half $s-$plane is the upper half $\omega-$plane. The motion becomes unstable when $\hat{f}_1$ acquires a singularity with Im $\omega > 0$. For current I less than some critical I_c, all singularities should be in lower half plane. At I_c a singularity hits the real axis, but at that point the factor $\omega - m\Omega(J)$ might vanish for some (m, J).

If ω is such that this factor vanishes, the equation is singular, a so-called *integral equation of the third kind.*

Integral Equation of the Third Kind:

$$a(x)\phi(x) + \int_0^1 K(x,y)\phi(y)dy = f(x) \ ,$$
$$a(x_0) = 0 \ , \quad x_0 \in (0,1) \ . \qquad (13)$$

1. G.R. Bart and R.W., SIAM J. Anal. **4** (1973) 609-622.

2. G.R. Bart, J. Math. and Appl. **79** (1981) 48-57.

3. V.S. Rogozhin and S.N. Raslambekov, Izvestiya VUZ. Matematika, **23** (1979) 61-69.

3rd-kind equations have *generalized function* solutions of the form

$$\phi(x) = \psi(x) + \lambda(P\frac{1}{x - x_0} + b\delta(x - x_0)) \ , \qquad (14)$$

where ψ is regular. If the kernel K is sufficiently smooth, and *the constant* b *is fixed*, then ψ satisfies a regular Fredholm equation (2nd kind), λ is determined, and the Fredholm alternative theorems hold for (13) in the space of generalized functions of form (14). For instance, the inhomogeneous equation has a unique solution if there is no solution of the homogeneous equation.

van Kampen considered solutions of the linear Vlasov for electrons in a uniform medium of positive charge (Coulomb force only). He admitted generalized functions of form (14), but *allowing* b *to vary*. Then the set of all such functions form a basis in which an arbitrary solution of Vlasov can be expanded.

This approach is not directly relevant to our problem. Rather, I argue that as ω tends to the real axis from above, a specific $b = -i\pi$ is picked out automatically.

For Im $\omega > 0$ put

$$\hat{f}_1(m, J, \omega) = \frac{g(m, J, \omega) e^{-J/2}}{\omega - m\Omega(J)} \ , \tag{15}$$

rewrite the equation with g as the unknown, and take the limit Im $\omega \to 0+$. Then for real ω,

$$\begin{aligned} g(m, J, \omega) &+ \\ \lim_{\epsilon \to 0+} \quad & \sum_{m'} \int dJ' H(m, J, m', J') \frac{g(m', J', \omega)}{\omega + i\epsilon - m'\Omega(J')} \\ = \quad & i\check{f}(m, J, 0) \ , \end{aligned} \tag{16}$$

where the kernel H decreases rapidly when either J or $J' \to \infty$:

$$H(m, J, m', J') = i e^{J/2} K(m, J, m', J') e^{-J'/2} \ . \tag{17}$$

If g has at least minimal smoothness then the limit in (16) exists, consisting of a PV part and a δ part for those cases in which the denominator vanishes at some m', J'.

See N.I. Muskhelishvili, “Singular Integral Equations”, for the proof that

$$\begin{aligned} &\lim_{\epsilon \to 0+} \int_a^b \frac{f(y)dy}{y - x + i\epsilon} \\ &= P \int_a^b \frac{f(y)dy}{y - x} - i\pi f(x) \ , \ x \in (a, b) \end{aligned} \tag{18}$$

provided that f is Hölder-continuous,

$$\begin{aligned} &|f(u) - f(v)| \le \kappa |u - v|^\alpha \ , \\ &u, v \in [a, b] \ , \quad 0 < \alpha \le 1 \ . \end{aligned} \tag{19}$$

For a given ω in (16), the denominator can vanish only for a few m', perhaps only one. Define

$$\mathcal{L}_\omega = \left[m \middle| \omega - m\Omega(J) = 0, \text{ for some } J \in [0, \infty) \right] , \tag{20}$$

i.e., the small set of modes, say $N(\omega)$ in number, for which the denominator can vanish. Denote its characteristic function by $\chi_\omega(m)$ which is 1 if $m \in \mathcal{L}_\omega$ and 0 otherwise. Also, let $\Omega^{-1}(\omega)$ be the inverse of $\Omega(J)$, and put

$$\lambda(m, \omega) = g(m, \Omega^{-1}(\omega/m), \omega) \ . \tag{21}$$

We can now evaluate the δ part of the integral in (16) by making a local change of integration variable to $y = \Omega(J')$. The equation takes the form

$$\begin{aligned} & g(m, J, \omega) + \\ & \sum_{m'} P \int dJ' H(m, J, m', J') \frac{g(m', J', \omega)}{\omega - m'\Omega(J')} + \\ & i\pi \sum_{m'} \frac{\chi_\omega(m')\lambda(m', \omega)}{|m'|\Omega'(\Omega^{-1}(\omega/m'))} H(m, J, m', \Omega^{-1}(\omega/m')) \\ & = i\check{f}(m, J, 0) \ . \end{aligned} \tag{22}$$

We write the equation (22) in abstract form as

$$g + Ag = i\check{f} \ . \tag{23}$$

It appears to me that the system (23) has good mathematical properties, provided that the wake is represented as a sufficiently smooth function. I am almost certain that A is a contractive operator on an easily described metric space, for sufficiently small current I. For instance, the space might be all g such that $\partial g/\partial J$ exists and is continuous, and the following quantity, identified as the norm, is finite:

$$\|g\| = \sup_{J,m} |m^2(|g(m,J)| + |\partial g/\partial J(m,J)|)| \ , \tag{24}$$

(or thereabouts).

Better yet, it appears likely that A is compact on some similar space, which means that it has the good properties of a Fredholm operator, even if it does not have the usual form of a Fredholm integral equation. In particular, a compact operator can be approximated by a finite dimensional operator (Riesz-Schauder theory). A compact operator is one that maps a bounded set into a relatively compact set. This requires some "smoothing and improving" action of the operator, which in our case should come from the fact that the J and m dependence of Ax comes entirely from the good kernel function $H(m, J, \cdots)$ which falls off rapidly in J and m and even has a number of derivatives in J (depending on the smoothness of the wake).

If A is indeed compact, we can justify replacing (23) by a finite dimensional matrix equation. With the same notation for the finite-D problem, our goal is then to find zeros of the determinant on the real $\omega-$axis,

$$\det(1 + A(\omega, I)) = 0 \ , \tag{25}$$

which should be absent for $I < I_c$ and first present at $I = I_c$.

The specific route to a finite-D problem will require some study. I describe a tentative scheme based on finitely many unknowns $g(m, J_j)$, (their argument ω being suppressed), where $\{J_j\}$ is a finite mesh in J-space, and $|m| < M$. For m' not in $\mathcal{L}_\omega$ we replace the integral of (16) by a numerical integration by some quadrature rule with weight w_j for mesh point J_j, thereby by getting a linear sum over unknowns $g(m', J_j)$. For $m' \in \mathcal{L}_\omega$, let J_i be the mesh point closest to $\Omega^{-1}(\omega/m')$, and write the integral for $J' \in [J_{i-1}, J_{i+1}]$ as follows:

$$\int_{J_{i-1}}^{J_{i+1}} dJ' \frac{H(m, J, m', J') - H(m, J, m', \Omega^{-1}(\omega/m'))}{\omega - m'\Omega(J')} g(m', J')$$
$$+ H(m, J, m', \Omega^{-1}(\omega/m')) \int dJ' \frac{g(m', J')}{\omega + i0 - m'\Omega(J')} \ . \tag{26}$$

The rest of the integral on J' and the first term of (26) can be treated by plain numerical quadrature as mentioned above. To make sure that the divided difference in the first term of (26) is indeed regular at the zero of the denominator, we need accurate consistency of the approximations for Ω and Ω^{-1}. I suggest that we replace $\Omega(J)$ by its quadratic Taylor approximation about J_i. The inverse of that quadratic function can be found analytically, thus giving perfect consistency. Since we anticipate a fairly fine mesh, and $\Omega(J)$ is typically an uneventful function, this should work nicely.

Furthermore, we can use the quadratic approximation of Ω in the second term, together with a representation of $g(m', J')$ as a quadratic function of J' interpolating the values $g(m', J_{i-1}), g(m', J_i), g(m', J_{i+1})$. The PV integral and δ term can then be evaluated analytically, to give a linear combination of those three values of g. Letting J run over mesh values, we thus get the finite-D matrix A. Note that the "pole subtraction" of (26) will account for a very small part of computer time, since it affects the matrix elements only for the *small* set of column indices $(m', J_j), \quad j = i-1, i, i+1$.

The canonical transform can be obtained in the form

$$Q(\phi, J) = \sum_m q_m(J) e^{im\phi} \ , \tag{27}$$

where the coefficients are represented as spline functions (of arbitrary smoothness). Similarly, $\Omega(J)$ and $\Omega^{-1}(\omega)$ can be given as spline functions. All that this requires is multiple numerical integrations of the equations of motion in the distorted well. Of course, we have to solve Haïssinski and redetermine the canonical transform for each current I.

A simpler possibility:

Solve the equation as a function of u for $\omega = u + iv$, with fixed v positive and small. Look for the first zero of the determinant as I is increased, thereby getting a lower bound on I_c. If v were some reasonable fraction of the synchrotron frequency ω_s, (let's say 1/100) then the denominator $u + iv - m\Omega(J)$ might be large enough in magnitude to avoid any numerical problem, and certainly the integral equation is Fredholm. Then the resulting lower . bound on I_c is the current at which a perturbation grows by a factor of e in a time of $100/2\pi$ synchrotron periods, which may be close enough to I_c for practical purposes.

Summary:

It appears that the simple change of unknown function given in (15) results in a good mathematical formulation of the linearized Vlasov problem, based on a compact linear operator. A rigorous proof will be attempted. Compactness of the operator will justify approximation of the equation by a finite-D system, whereas no such approximation can be justified for the equation in its original form. Codes based on the original equation have had some qualified success (Oide et al.), but there are reports that one does not get convincing convergence as the mesh in J is refined, as I would expect.