Radiation of an ultrarelativistic particle passing through a round hole in a perfectly conducting screen

G. Stupakov

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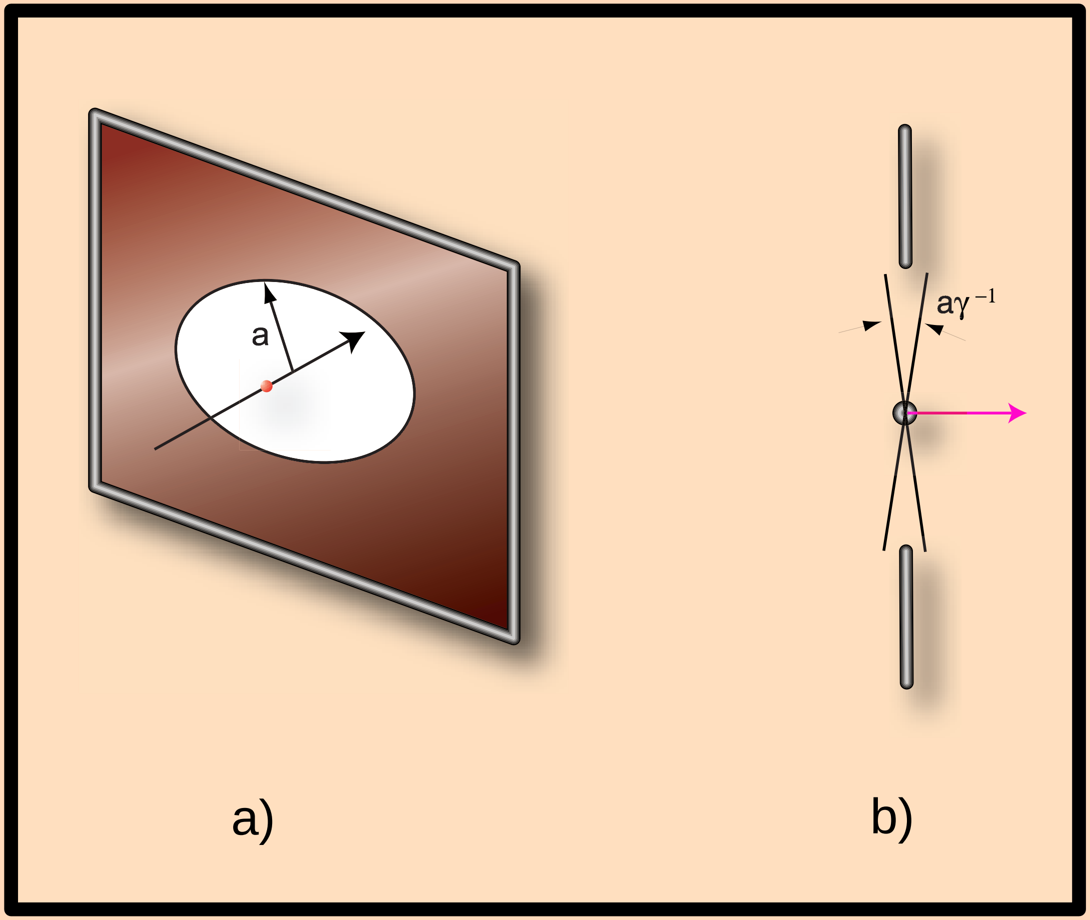
Radiation of charge...



A charge passing through a round hole in a metal screen.

Radiation of charge...

Previous papers

- Dnestrovskii and Kostomarov, 1959
- Dome et al., 1991

$$\int_0^\infty uF(u,k)J_1(ur)du = 0 \qquad 0 \le r < R_0; \qquad (2)$$

$$\int_0^\infty u\sqrt{u^2-k^2}F(u,k)J_1(ur)du = AK_1(\alpha r) \qquad r > R_0, \qquad (3)$$

where A and α are respectively

$$A = \frac{iqk^2}{\pi\beta^2\gamma} \quad \text{and} \quad \alpha = \frac{k}{\beta\gamma},$$

Ultrarelativistic case

$F \to AF$ – this removes A from eqs.

$k \simeq \alpha\gamma$ $(\beta \approx 1)$

$\alpha \sim 1$

- Bolotovskii and Galst'yan, 2000

The field of relativistic charge q, moving with a constant velocity along a straight line with the Lorentz factor γ is

$$E_r = H_\theta = \frac{q\gamma r}{[r^2 + \gamma^2 (z - ct)^2]^{3/2}}$$

Take this field at $z = 0$ and Fourier transform it:

$$\mathcal{E}_r(r, \omega) = \mathcal{H}_\theta(r, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dte^{i\omega t} E_r(r, t)|_{z=0} = \frac{kq}{\pi c\gamma} K_1 \left(\frac{kr}{\gamma} \right) .$$

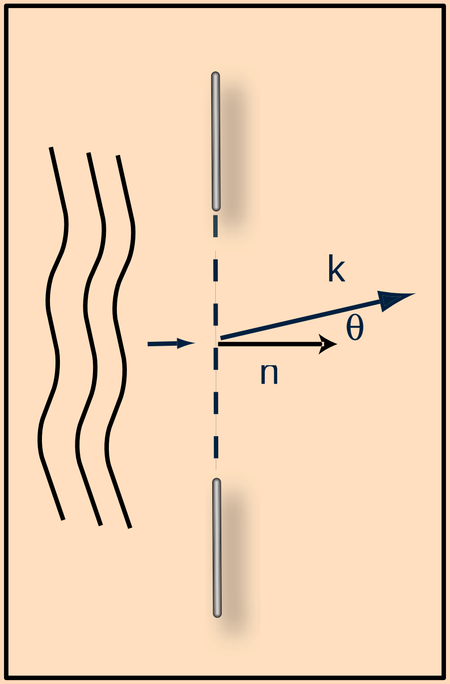
This is the field in the plane of the hole.

This field generates currents in the metal and results in radiation. From the symmetry of the problem, the radiation is symmetric:

$$\boldsymbol{E} = \boldsymbol{E}^{\text{beam}} + \boldsymbol{E}^{\text{rad}} , \qquad \boldsymbol{H} = \boldsymbol{H}^{\text{beam}} + \boldsymbol{H}^{\text{rad}} ,$$

with E_x^{rad}, E_y^{rad}, H_z^{rad} even, and E_z^{rad}, H_x^{rad}, H_y^{rad} odd functions of z.

Radiation of charge...



Diffraction formulae from Jackson's "Classical Electrodynamics". The field in the far zone, $\boldsymbol{\mathcal{E}}^{\rm rad}$, at distance $R \to \infty$ from the hole

$$\boldsymbol{\mathcal{E}}^{\rm rad} = \frac{e^{ikR}}{R}\boldsymbol{F}(\boldsymbol{k}),$$

where

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{k}) &= \frac{i}{4\pi}\int_S dS \exp(-i\boldsymbol{k}\boldsymbol{r})\,[k(\boldsymbol{n}\times\boldsymbol{\mathcal{H}}) + \boldsymbol{k}\times(\boldsymbol{n}\times\boldsymbol{\mathcal{E}}) - \boldsymbol{k}(\boldsymbol{n}\cdot\boldsymbol{\mathcal{E}})] \\ &= \frac{1}{4\pi i}\boldsymbol{k}\times\int_S dS \exp(-i\boldsymbol{k}\boldsymbol{r})\left[\frac{\boldsymbol{k}}{k}\times(\boldsymbol{n}\times\boldsymbol{\mathcal{H}}) - \boldsymbol{n}\times\boldsymbol{\mathcal{E}}\right],\end{aligned}$$

with $\boldsymbol{r}$ the two-dimensional vector in the plane of the hole, and $\boldsymbol{k}$ – the wavenumber vector in the direction of the radiation.

We use diffraction formulae with source – beam field.

Assume that the angle of the radiation relative to the z axis, θ, is small, $\theta \ll 1$. Then $\boldsymbol{F}$ has the radial component only

$$F_r = -k \int_0^a r dr \mathcal{E}_r(r,\omega) J_1(kr\theta) = -\frac{qk^2}{\pi\gamma c} \int_0^a r dr K_1\left(\frac{ka}{\gamma}\right) J_1(kr\theta)$$

This field includes both the vacuum beam field and the radiation field. To get the radiation field only, subtract the beam field given by the limit $a \longrightarrow \infty$.

$$F_r \to \frac{qk^2}{\pi\gamma c} \int_a^\infty r dr K_1\left(\frac{ka}{\gamma}\right) J_1(kr\theta)$$

Using a formula from Gradshtein and Ryzhik:

$$\mathcal{E}_r^{\mathrm{rad}} = -\frac{q}{\pi\gamma c}\frac{ka}{\theta^2+\gamma^{-2}}\left[\theta J_2(ka\theta)K_1\left(\frac{ka}{\gamma}\right) - \frac{1}{\gamma}J_1(ka\theta)K_2\left(\frac{ka}{\gamma}\right)\right]\frac{e^{ikR}}{R}$$

Radiated energy $U_{\rm rad}$:

$$\begin{aligned} U_{\rm rad} &= \int_{-\infty}^{\infty} dt \int_S dS \frac{c}{4\pi} (E_r^{\rm rad})^2 \\ &= c \int_{-\infty}^{\infty} d\omega \int_S dS |\mathcal{E}_r^{\rm rad}|^2 \\ &= c \int_{-\infty}^{\infty} d\omega \int_\Omega d\Omega |F_r^{\rm rad}|^2 \approx c \int_{-\infty}^{\infty} d\omega \int_0^{\infty} \theta d\theta |F_r^{\rm rad}|^2 \\ &= \int_0^{\infty} d\omega P(\omega) \end{aligned}$$

with the spectral energy:

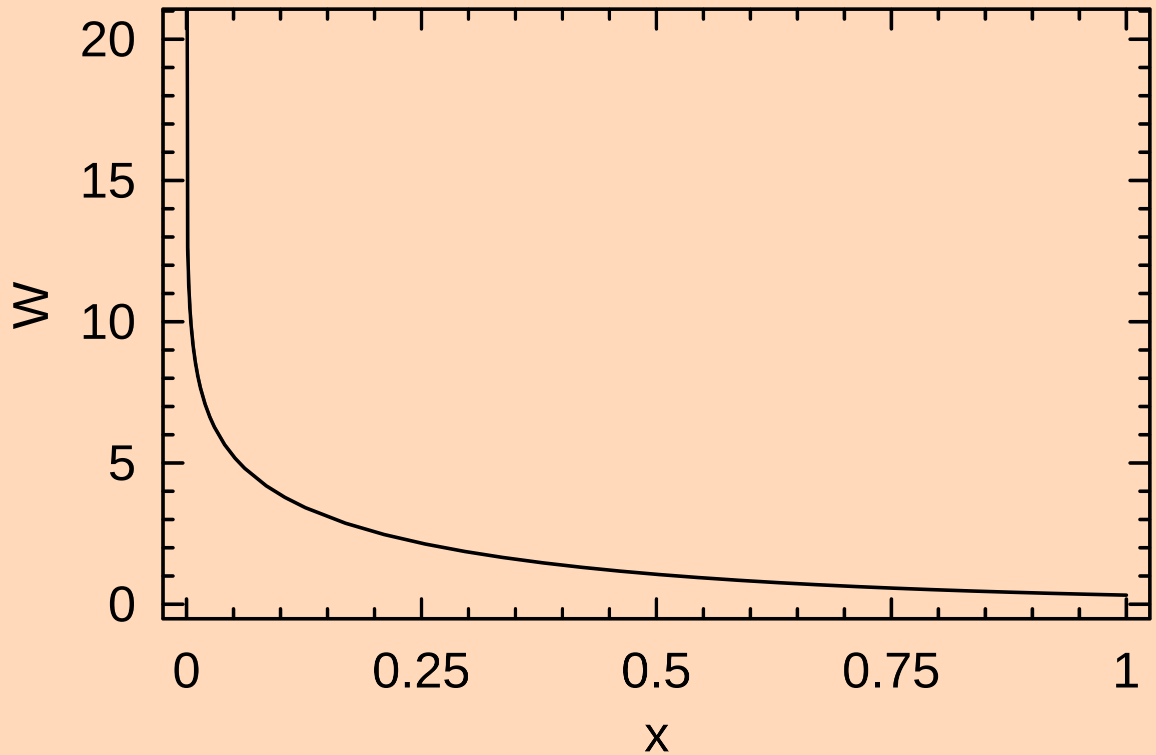
$$P(\omega) = \frac{2}{\pi} \frac{q^2}{c} W\left(\frac{ak}{\gamma}\right),$$

where

$$W(x) = x^2 \left[K_0(x) K_2(x) - K_1(x)^2\right]$$

with K_n – the modified Bessel functions of the second kind. The function W has a logarithmic singularity at $x = 0$.

Radiation of charge...



To find the total radiated energy, we integrate P over the frequency

$$\int_0^\infty P(\omega)d\,\omega = \frac{3\pi}{8}\frac{q^2\gamma}{a}\,.$$

We can compare this with the energy U of the electromagnetic field that is “clipped” by the iris. The fields of the ultrarelativistic charge are

$$E_r = H_\theta = \frac{\gamma q r}{(r^2+\gamma^2 z^2)^{3/2}}\,,$$

and the energy density w

$$w = \frac{1}{8\pi}(E_r^2 + H_\theta^2)\,.$$

Integrating w over the region $r > b$ and over z yields

$$U = \int_b^\infty 2\pi\, r\, dr \int_{-\infty}^\infty dz\, w = \frac{3\pi}{16}\frac{q^2\gamma}{a}\,.$$

We see that the radiated energy is equal to twice the clipped energy. This happens because the clipped field is reflected back by the screen, and is radiated in the backward direction. The same amount of energy is radiated in the forward direction when the charge grows the new field.

We can relate the real part of the longitudinal impedance to the radiated power using the formula

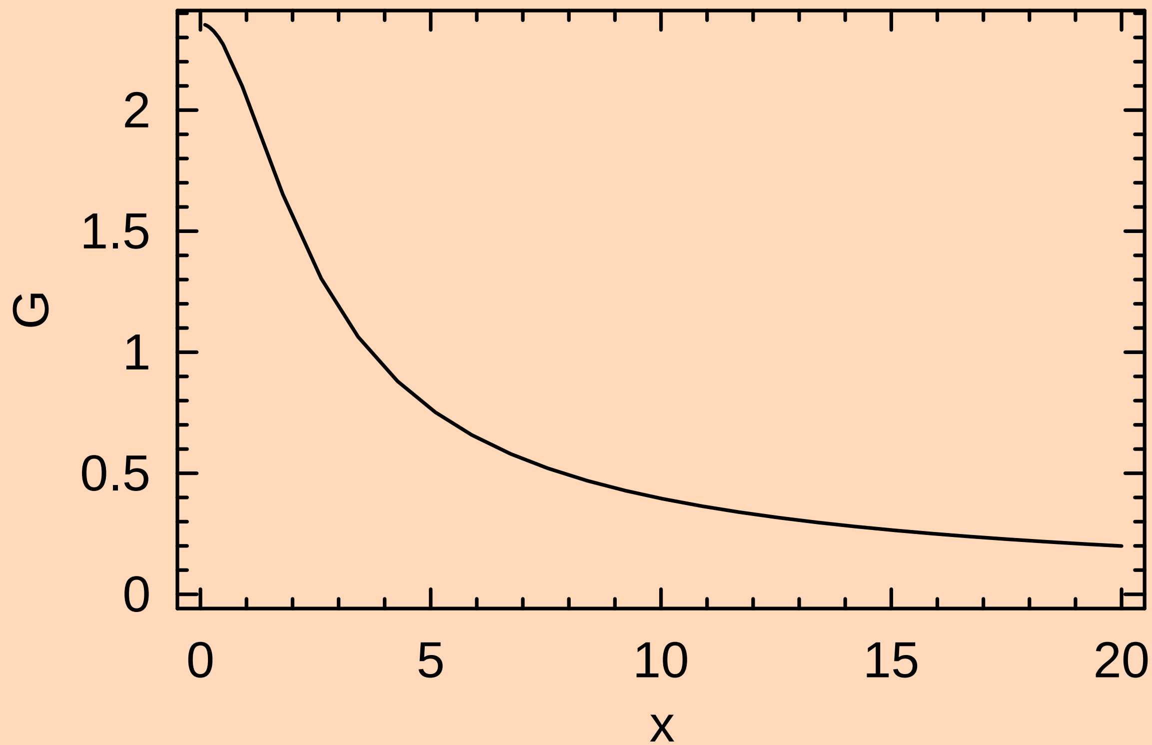
$$\mathrm{Re}\, Z(\omega) = \frac{\pi}{q^2} P(\omega) = \frac{2}{c} W\left(\frac{a\omega}{c\gamma}\right) ,$$

We can calculate the wake using the relation (can we?)

$$w(s) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\, Z(\omega)\cos\left(\frac{\omega s}{c}\right) d\omega = \frac{\gamma}{a} G\left(\frac{\gamma s}{a}\right) ,$$

where

$$G(\xi) = \frac{4}{\pi}\int_0^\infty F(x)\cos\left(x\xi\right) d\omega \,.$$



G
0
0.5
1
1.5
2
0
5
10
15
20
x

Radiation of charge...

Comparison with other papers

1. Full agreement with Dome et al.
2. Dnestrovskii and Kostomarov:

$$H_\varphi(R, \vartheta, \omega) = -\frac{2e}{c^2}\frac{\gamma|\omega|a}{c}K_1\left(\frac{\gamma|\omega|a}{c}\right)J_0\left(\frac{\omega a}{c}\sin\vartheta\right)\frac{\sin\theta}{\gamma^2+\sin^2\vartheta}\frac{e^{i\omega\frac{R}{c}}}{R},$$

3. Bolotovskii and Galst'yan:

Does not agree with Dome

The magnetic field has a single nonzero component H_φ:

$$H_\varphi = \frac{q}{\pi}\frac{v^2}{c^3}\frac{\exp(ikr)}{r}\frac{1}{1-(v^2/c^2)\cos^2\theta}$$
$$\times\left[a\frac{\omega}{c}\sin\theta J_1\left(a\frac{\omega}{c}\sin\theta\right)K_0\left(a\frac{\omega}{v}\sqrt{1-\beta^2}\right)\right.$$
$$\left.-a\frac{\omega}{v}\sqrt{1-\beta^2}J_0\left(a\frac{\omega}{c}\sin\theta\right)K_1\left(a\frac{\omega}{v}\sqrt{1-\beta^2}\right)\right]\sin\theta\cos\theta. \quad (59)$$