Review of Feedback Basics

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1 Abstract

Basics of the feedback theory are presented.
2 Plan:

- Basic feedback:
  closed and open loop transfer functions,
  Nyquist criterion of stability.
- Longitudinal feedback:
  cavity impedance,
  steady state conditions, optimum conditions,
  F.Pedersen’s analysis
- Actual longitudinal feedbacks
- Transverse feedback.
3 Transfer functions of control system

Dynamics of a linear system is described by a system of linear diff. equations \( \hat{L}_n(t)y(t) = U(t) \) of the n-th order,

\[
\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum b_k \frac{d^k u(t)}{dt^k},
\]

where \( a_k, b_k \) are constants.

Laplace transform \( y(t) \rightarrow \tilde{y}(s), y(s) = \int_0^{\infty} f(t)e^{-st}, \)

\[
y(t) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{ds}{2\pi i} \tilde{y}(s)e^{st},
\]

gives for driven solution \( \tilde{y}(s) = G_n(s)\tilde{u}(s) \), where \( G_n(s) \) is the transfer function,

\[
G(s) = \frac{\sum b_k s^k}{\sum_{k=0}^{n} a_k s^k} = |G(s)|e^{i\Phi(s)},
\]

and the denominator is characteristic polynomial.
Single element: $T(s)$

Cascade (open loop transfer function):

$$a \rightarrow b = Ga; b \rightarrow c = Hb = GHa; T = c/a = GH;$$

Parallel elements: $T(s) = T_1(s) + T_2(s)$.

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<td>2 Combining Blocks in Parallel; or Eliminating a Forward Loop</td>
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<td>$X \rightarrow P_1 \rightarrow - \rightarrow P_2 \rightarrow Y$</td>
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Figure 1: Transfer function of a control system.
Feedback transfer function:

\[ E = R - B; \quad C = GE; \quad B = HC = HGE; \]
\[ E = R - HGE; \quad E = \frac{R}{1 + HG}; \quad T = \frac{C}{R} = \frac{G}{1 + HG} \]

Note: charact. polinomial \( GH \) is the open loop TF.
Note: Discrete feedback system:

\[ y_n + b y_{n-1} = a_0 x_n + a_1 x_{n-1}. \]

\( b = 0, \) finite impulse response filter (FIR),
\( b \neq 0, \) infinite response filter (IIR). Stable if \( |b| < 1. \)
3.1 Conditions of stability

The characteristic polynomial is

\[ P_n(s) = \sum_{k=0}^{n} a_k s^k. \]

The eigen-values \( s_k, k = 0, 1, \ldots, n \) are the roots \( P_n(s) = 0 \).

Condition of stability \( \text{Re}[s_k] < 0 \).

Routh and Hurwitz conditions of stability: rules on combinations of coefficients \( a_k \).

Nyquist stability criterion:

(don’t confuse with the Nyquist criterion of sampling: sampling rate \( \frac{1}{\Delta t} > 2\Delta f \)):

Consider the transfer function in the form:

\[ T(s) = \frac{1}{1 + GH(s)} \], where \( GH(s) \) is open loop transfer function.
Use theorem: for $f(s) = 1 + GH(s)$, the integral

$$J = \frac{1}{2\pi i} \int_C \frac{f'(s)}{f(s)} ds;$$

$$J = P - Z = n_f(0) = n_{GH}(-1)$$

where $P$ and $Z$ are the number of poles/zeros within the contour $C$ in $s$-plane, and $n_f(0)$ number of loops around the origin by the map $s \rightarrow f(s) = 1 + GH(s)$, or, the number of loops around $-1$ by $GH(s)$ in $Re[GH], Im[GH]$ plane.

**Criterion:**

If $n_{GH}(-1) = P$, there are no zeros of $f(s)$, i.e. no poles of $T(s)$ with $Re(s) > 0$ and the system is stable.
Figure 3: Nyquist criterion for transfer function $G(s)$: upper plot: $n = P = 0$, stable. Lower plot: $n = -1$, $P = 1$, unstable.
4 Low level feedback

\[ V_c = Z_c (I_g - I_{FB}), \quad I_{FB} = \frac{V_c}{Z_A} = \frac{Z_c (I_g - I_{FB})}{Z_A}, \]

\[ I_{FB} = \frac{Z_c}{Z_A} \frac{I_g}{1 + Z_g / Z_A}, \quad V_c = \frac{I_g Z_c}{1 + Z_c / Z_A} \]

Figure 4: Basic direct loop of the cavity FB.
4.1 Cavity impedance

Impedance of a cavity including coupling $\propto \beta$ is given in terms of the loaded shunt impedance $R_L$ and loaded $Q_L$-factor:

$$Z_c(\omega) = \frac{R_L}{1 + iQ_L(\omega_c/\omega - \omega/\omega_c)} \approx \frac{R_L}{1 - 2iQ_L \frac{\omega - \omega_c}{\omega_c}},$$

$$R_L = \frac{R_0}{1 + \beta}, \quad R_L = \frac{R_0}{1 + \beta}, \quad Q_L = \frac{Q_0}{1 + \beta}.$$  

At $\omega = \omega_g$,

$$Z_c(\omega_g) = R_L \cos \psi e^{i\psi},$$

$$\tan \psi = Q_L \left( \frac{\omega_g}{\omega_c} - \frac{\omega_c}{\omega_g} \right).$$
4.2 Low-level FB transfer function

The effective impedance of the cavity with the direct FB:

\[ V_c(\omega) = \frac{I_g Z_c}{1 + Z_c/Z_A} = \frac{I_g R_L}{1 - 2i Q_L \frac{\omega - \omega_c}{\omega_c}} + G_{FB}(\omega). \]

\[ G_{FB}(\omega) = \frac{R_L}{Z_A} = H(\omega) e^{-i \Phi(\omega)} \approx H(\omega_c) e^{-i (\omega - \omega_c) \tau_d}, \]
\[ \approx H_g (1 - i (\omega - \omega_c) \tau_d) \]
Hence,

\[ \begin{align*}
Z_{eff} &= \frac{R_H}{1 - 2iQ_H(\omega - \omega_c)/\omega_c}, \\
R_H &= \frac{R_L}{1 + H_g}, \quad Q_H = \frac{Q_L}{1 + H_g} + \frac{H_g}{1 + H_g} \frac{\omega_g \tau_d}{2}.
\end{align*} \]

FB reduces \( R_L \) but increases the bandwidth of revolution harmonics affected by the cavity impedance.

Numbers: \( Q_0 = 3.0 \times 10^4, \quad Q_L = 6.6 \times 10^3, \quad H_g \sim 12, \quad \tau_d = (380 + 150) \) ns, \( \omega_g \tau_d/2 \approx 700. \)
Figure 5: Phase notations.

\[ I_{\text{tot}} = I_g - I_b \]
4.3 Steady state of rf cavity: $\omega = \omega_g$, FB off

Total excitation current for a cavity: $I_{tot} = I_g - I_B$.

Define:

$$I_{beam}(t) = 2I_{dc} \cos(\omega_gt + \phi_c - \phi_s) = \frac{1}{2}(\hat{I}_b e^{-i\omega_g t} + c.c.),$$

$$\hat{I}_b = 2I_{dc}e^{i(\phi_c - \phi_s)}, \quad \hat{I}_b = I_{beam}(\omega_g),$$

$$V(t) = V_{cav} \cos(\omega_gt + \phi_c) = \frac{1}{2}(\hat{V}_c e^{-i\omega_g t} + c.c.),$$

$$\hat{V}_c = V_{cav}e^{-i\phi_c}, \quad \hat{V}_c = V_{cav}(\omega_g)$$

$$I_{gen}(t) = I_g \cos(\omega_gt) = \frac{1}{2}(\hat{I}_g e^{-i\omega_g t} + c.c.),$$

$$\hat{I}_g = I_g, \quad \hat{I}_g = I_{gen}(\omega_g).$$

$I_{dc}$ is the average beam current, $V_{cav}$ is maximum cavity voltage.

Then, defining $\hat{I}_{tot} = \frac{1}{2}(\hat{I}_{tot} e^{-i\omega_g t} + c.c.),$

$$\hat{I}_{tot} = \hat{I}_g - |\hat{I}_b|e^{i(\phi_c - \phi_s)}$$

$$V_{cav} e^{-i\phi_c} = \hat{I}_{tot} R_L \cos \psi e^{i\psi}.$$
Separating real and imaginary parts,

\[ \hat{I}_g = |\hat{I}_B| \frac{\sin(\phi_s + \psi)}{\sin(\phi_c + \psi)} \]

\[ \frac{1 + \beta}{Y} = \cos \psi \frac{\sin(\phi_s - \phi_c)}{\sin(\phi_c + \psi)}, \]

where \( Y = \frac{2R_0 I_{dc}}{V_{cav}} \).

Accelerating voltage \( V_{acc} = V_{cav} \cos(\phi_s) \).

\( \beta \) is defined by the cavity design (matching with the wave guide), \( \beta \approx 3.5 \).

The phase \( \phi_s \) is defined by the energy loss/turn: \( V_c \cos(\phi_s) = U \).

The detuning angle \( \psi \) is given by
\( \tan \psi = 2Q_l(\omega_g - \omega_c)/\omega_c \).

### 4.3.1 Optimum conditions

Reflected power is zero provided \( \phi_c = 0 \) and \( \hat{I}_g = \frac{2\beta}{R_0} V_c \).

In this case

\[ \beta = 1 + Y \cos \phi_s, \quad \tan \psi = \frac{\beta - 1}{\beta + 1} \tan \phi_s. \]
Example: Some LER PEP-II parameters:

\[ E = 3.1 \, GeV; \quad f_0 = 136 \, KH\ell; \quad f_{rf} = 496 \, MH\ell; \]
\[ I_{dc} = 2.25 \, A; \quad V_{cav} = 0.85' \, MeV; \quad n_{cav} = 6.; \]
\[ N_b = 5.8125 \times 10^1 \, 0; \quad s_b = 124.' \, cm; \quad \phi_s = 80.9^0; \]
\[ \beta = 3.9; \quad Y = 18.5; \quad \tau_d = 450.0 \, ns; \quad \psi = 74.9^0; \]
\[ \Delta f = -145.1 \, KH\ell; \quad f_s = 4.5 \, KH\ell; \quad Q_H = 6111.1; \]
\[ U_{SR} = 0.76 \, MeV; \quad P_{forward/cavity} = 40.3 \, kW \]
4.4 Robinson condition of stability

Energy gain leads to the shift $z < 0$, and $I_{beam}$ get phase

\[ I_{beam}(t) = 2I_{dc} \cos(\omega t + \phi_c - \phi_s + \frac{\omega_g z}{c}). \]

Hence, $\phi_s$ changes, $\phi_s \rightarrow \phi_s - \frac{\omega_g z}{c}$.

$I_g$, $I_{dc}$, $\psi$, $\beta$ do not change. Therefore, $\phi_c$ and $Y$ must change. For stability, the accelerating voltage $V_{acc} \propto (1/Y) \cos \phi_s$ has to change

\[ d(\cos(\phi_s)/Y)d\phi_s < 0. \]

That, after some algebra, gives Robinson criteria of stability:

\[ 0 < \sin(2\psi) < 2\left(1 + \frac{\beta}{Y}\right) \sin \phi_s. \]
5 F. Pedersen’s analysis of stability

Small variation around the steady-state:

\[
\delta V_{cau}(t) = \frac{1}{2} \hat{V}_c e^{-i\omega_g t} [u(t) - i\Phi(t)],
\]

\[
\delta I_{tot}(t) = \frac{1}{2} \hat{I}_{tot} e^{-i\omega_g t} [a(t) - ip(t)],
\]

where \(u(t), \Phi(t), a(t), p(t)\) are slow functions of time.

Take Fourier transform: \(a(t) \rightarrow a(\omega)\),
\(a(t)e^{-i\omega_g t} \rightarrow a(\omega - \omega_g)\).

\[
\delta V_c(\omega) = \frac{1}{2} \hat{V}_c [u(\omega - \omega_g) - i\Phi(\omega - \omega_g)],
\]

\[
\delta I_{tot}(\omega) = \frac{1}{2} \hat{I}_{tot} [a(\omega - \omega_g) - ip(\omega - \omega_g)].
\]
Use $\hat{V}_c = Z_c(\omega_g)\hat{I}_{tot}$, $\delta V_c(\omega) = Z_c(\omega)\delta I_{tot}(\omega)$. Then,

$$u(\omega) - i\Phi(\omega) = \frac{Z_c(\omega + \omega_g)}{Z_c(\omega_g)} [a(\omega) - ip(\omega)].$$

In F. Pedersen notations (in Laplace transform), the transfer functions are:

$$\tilde{u}(s) = G_{aa}\tilde{a}(s) + G_{ap}\tilde{p}(s),$$
$$\tilde{\Phi}(s) = G_{pa}\tilde{a}(s) + G_{pp}\tilde{p}(s),$$

where

$$G_{aa} = G_{pp} = Re\left[\frac{\tilde{Z}_c(s - i\omega_g)}{\tilde{Z}_c(-i\omega_g)}\right],$$
$$G_{ap} = -G_{pa} = Im\left[\frac{\tilde{Z}_c(s - i\omega_g)}{\tilde{Z}_c(-i\omega_g)}\right].$$
To close the system, define

\[ \delta I_{tot}(t) = \delta I_{gen}(t) - \delta I_{FB}(t) - \delta I_{B}(t). \]

The beam current at \( s_{cav} \)

\[
I_{beam}(t) = \frac{I_{dc}}{n_{b}cT_{0}} \sum_{n=1}^{n_{b}} \delta(ct - ns_{b} + \zeta_{n} - s_{cav})
\]

\[
= \frac{I_{dc}}{n_{b}} \sum_{n} \sum_{k} e^{\frac{2\pi ik}{c}(ct-ns_{b}+\zeta_{n}-s_{cav})}
\]

Retain harmonics \( k = \pm n_{b} \) and put \( \zeta_{n} = z(t) \), (i.e. \( m = 1 \) mode) and \( s_{cav} = 0 \):

\[
I_{beam}(t) = I_{dc}e^{-i(\omega_{g}t+\frac{\omega_{g}\zeta}{c})} + c.c.
\]

In the linear approximation over \( \zeta \),

\[
\delta I_{b}(t) = -iI_{dc}e^{-i\omega_{g}t} \left( \frac{\omega_{g}\zeta(t)}{c} \right) + c.c.
\]
Consider the beam dynamics with $n_{cav}$ cavities in the ring:
\[
\frac{d^2 \zeta(t)}{dt^2} + \omega_s^2 \zeta(t) = -\frac{\alpha c \omega_0 n_{cav} e \delta V_c(t)}{2\pi E}.
\]

Note, $\delta V_c$ is taken at the arrival time $t_b = \{(\phi_s - \phi_c)/\omega_g - \zeta(t_b)/c\} \mod [T_{rev}]$ and here it can be written as $\delta V_{cav}(t) = \frac{1}{2} \hat{V}_c [u(t) - i\Phi(t)]$. That makes certain that $\zeta(t)$ is slow function of $t$. In frequency domain:
\[
\zeta(\omega) = -\frac{\alpha c \omega_0 n_{cav}}{4\pi E} \frac{e \hat{V}_c[u(\omega) - i\Phi(\omega)]}{\omega_s^2 - \omega^2}.
\]

For $\omega \simeq \omega_g$,
\[
\delta I_{beam}(\omega) = i \frac{\alpha \omega_g n_{cav} \omega_0}{4\pi E} \frac{e \hat{V}_c I_{dc}[u(\omega - \omega_g) - i\Phi(\omega - \omega_g)]}{\omega_s^2 - (\omega - \omega_g)^2}.
\]

The Fourier component of the FB current
\[
\delta I_{FB}(\omega) = G_{FB}(\omega) \delta V_c(\omega),
\]
\[
R_L G_{FB}(\omega) = H_g[1 - i(\omega - \omega_c)\tau_d].
\]
Using $\hat{V}_c = Z_c(\omega_g)\hat{I}_{tot}$ and the definition

$$\delta I_{tot}(\omega) = \frac{1}{2}\hat{I}_{tot}[a(\omega - \omega_g) - ip(\omega - \omega_g)],$$
$$= \delta I_{gen}(\omega) - \delta I_{FB}(\omega) - \delta I_B(\omega),$$

we get

$$a(\omega) - ip(\omega) = \frac{2\delta I_{gen}(\omega + \omega_g)}{\hat{I}_{tot}}$$
$$- \frac{H_g}{R_L} [1 - i(\omega + \omega_g - \omega_c)\tau_d] Z_c(\omega_g) [u(\omega) - i\Phi(\omega)]$$
$$- i\frac{\omega_s^2}{2\sin \phi_s} \frac{Z_c(\omega_g)|\hat{I}_b|}{V_{cav}} \frac{[u(\omega) - i\Phi(\omega)]}{\omega_s^2 - \omega^2}.$$
That can be written using $\hat{I}_b = 2I_{dc}e^{i(\phi_s-\phi_c)}$, and $Y = \frac{2R_0I_{dc}}{V_{cav}}$ in the form

\[
\begin{align*}
    a(\omega) - ip(\omega) &= \frac{2\delta I_{gen}(\omega + \omega_g)}{\hat{I}_{tot}} - T(\omega) [u(\omega) - i\Phi(\omega)], \\
    T(\omega) &= \cos(\psi)e^{i\psi} [H_g (1 - i(\omega + \omega_g - \omega_c)\tau_d) \\
    &+ \frac{i\omega_s^2 Y}{2(1 + \beta) \sin \phi_s} \frac{e^{i(\phi_s-\phi_c)}}{\omega_s^2 - \omega^2},
\end{align*}
\]

where, for $n_{cav}$ cavities

\[
\omega_s^2 = \frac{\alpha \omega_g \omega_0 eV_{cav} n_{cav} \sin(\phi_s)}{2\pi E}.
\]

The system of homogeneous equations

\[
\begin{align*}
    a(\omega) - ip(\omega) &= -T(\omega) [u(\omega) - i\Phi(\omega)], \\
    u(\omega) - i\Phi(\omega) &= \frac{Z_c(\omega + \omega_g)}{Z_c(\omega_g)} [a(\omega) - ip(\omega)].
\end{align*}
\]

define eigenvalues $\omega$. The system is stable if all eigenvalues are in the lower half plane of $\omega$. 

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The system can be written as the cubic equation for $x = \omega/\omega_s$:

$$(1-2ix \frac{\omega_s Q_H}{\omega_g} - i \frac{Q_H}{Q_L} \tan \psi) (x^2-1) = \frac{iY}{2(1 + \beta)(1 + H_g) \sin(\phi_s)}$$

As it was shown above, parameters $Y$ and $\phi_c$ are related,

$$\frac{1 + \beta}{Y} = \frac{\sin(\phi_s - \phi_c)}{\cos \psi \sin(\phi_c + \psi)}.$$
Figure 6: Stability diagram (F. Pedersen).
Figure 7: Stability diagram, see equation in the text.

Figure 8: Growth rate vs FB gain $H_g$. 
6 Full longitudinal FBs

Figure 9: Full longitudinal feedback system.
6.1 Comb filter

Figure 10: Real part of the transfer function of the comb filter.
6.2 Other loops

*Tuner loop*: minimize reflected power

*Klystron saturation loop*: control of the klystron operating point

*ripple loop*: adjust modulator to maintain constant gain/phase shift through modulator/klystron system

*Gap FB loop*: removes revolution harmonics from FB error signal to avoid saturation of the klystron

*Woofer link*: use the cavity as a kicker for low frequency modes
6.3 Noise of the FB

For longitudinal FB, see S.H. +A. Chao

7 Bunch-by-bunch longitudinal FB

Figure 11: Schematic of the bunch-by-bunch longitudinal FB.
Figure 12: Diagnostics with the bunch-by-bunch longitudinal FB.
8 Transverse FB

Figure 13: Schematics of the transverse FB.

However: Gproto may eliminate the DSP