Transverse to Longitudinal Emittance Exchange

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- Idea proposed by M. Cornacchia (Nov. 2001)
- Analysis taken from similar work by W. Spence and PE (unpublished, ~1990)
- Motivation to reduce transverse emittance and increase longitudinal emittance — faster SASE X-ray FEL lasing and less CSR micro-bunching in compressors

X-ray SASE FEL needs emittance…

$$\gamma\varepsilon_x < \gamma\frac{\lambda_r}{4\pi},$$

and energy spread…

$$\sigma_\delta < \rho \approx \frac{1}{4}\left(\frac{1}{2\pi^2}\frac{I_{pk}}{I_A}\frac{\lambda_u^2}{\beta\varepsilon_N}\left(\frac{K}{\gamma}\right)^2\right)^{1/3}.$$

At $\lambda_r \approx 1$ Å and $\gamma \approx 3\times10^4$ $\rightarrow$ $\boldsymbol{\gamma\varepsilon_x < 0.2\ \mu\mathrm{m}}$

But $\sigma_\delta < 0.09\%$ (for $I_{pk} \approx 4$ kA, $\lambda_u \approx 3$ cm, $K \approx 4$, $\beta \approx 20$)

This means $\gamma\sigma_z\sigma_\delta = \boldsymbol{\gamma\varepsilon_z < 500\ \mu\mathrm{m}}$

RF-guns produce $\gamma\varepsilon_x \sim 5\ \mu$m and $\gamma\varepsilon_z \sim 2\ \mu$m (and highly variable)

Can we exchange these in some way, reducing ε_x while increasing ε_z (maybe also damping the CSR microbunching) ?

Initial uncoupled 4×4 beam covariance matrix $\boldsymbol{\sigma_0}$ [$\gamma_x = (1+\alpha_x^2)/\beta_x$, $\gamma_z = (1+\alpha_z^2)/\beta_z$]…

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} \varepsilon_{x_0}\beta_x & -\varepsilon_{x_0}\alpha_x & 0 & 0 \\ -\varepsilon_{x_0}\alpha_x & \varepsilon_{x_0}\gamma_x & 0 & 0 \\ 0 & 0 & \varepsilon_{z_0}\beta_z & -\varepsilon_{z_0}\alpha_z \\ 0 & 0 & -\varepsilon_{z_0}\alpha_z & \varepsilon_{z_0}\gamma_z \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\sigma}_x & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_z \end{pmatrix}$$

$$\sigma_x = \sqrt{\varepsilon_{x_0}\beta_x}, \;\; \sigma_z = \sqrt{\varepsilon_{z_0}\beta_z}$$

$$\text{chirp:} \;\; \frac{\langle z\delta \rangle}{\sigma_z^2} = -\frac{\alpha_z}{\beta_z}, \qquad (\delta \equiv \Delta E / E_0)$$

$$E\text{-spread:} \;\; \sigma_\delta = \sqrt{\varepsilon_z\left(1+\alpha_z^2\right)/\beta_z} = \sqrt{\sigma_{\delta_u}^2 + \sigma_{\delta_c}^2}$$

$$\gamma\varepsilon_z = \gamma\sqrt{\sigma_z^2\sigma_\delta^2 - \langle z\delta \rangle^2}$$

$$\gamma\varepsilon_z\left(\alpha_z = 0\right) = \gamma\sigma_z\sigma_\delta = \sigma_z\frac{\sigma_E}{mc^2}$$

Propagate $\boldsymbol{\sigma_0}$ through 4×4 transfer matrix, **R**

$$\boldsymbol{\sigma} = \mathbf{R}\boldsymbol{\sigma}_0\mathbf{R}^T,$$

which is four 2×2 blocks

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad etc.$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{A}\boldsymbol{\sigma}_x\mathbf{A}^T + \mathbf{B}\boldsymbol{\sigma}_z\mathbf{B}^T & \mathbf{A}\boldsymbol{\sigma}_x\mathbf{C}^T + \mathbf{B}\boldsymbol{\sigma}_z\mathbf{D}^T \\ \mathbf{C}\boldsymbol{\sigma}_x\mathbf{A}^T + \mathbf{D}\boldsymbol{\sigma}_z\mathbf{B}^T & \mathbf{C}\boldsymbol{\sigma}_x\mathbf{C}^T + \mathbf{D}\boldsymbol{\sigma}_z\mathbf{D}^T \end{pmatrix}$$

$$\varepsilon_x^2 = \left| \mathbf{A}\boldsymbol{\sigma}_x\mathbf{A}^T + \mathbf{B}\boldsymbol{\sigma}_z\mathbf{B}^T \right|$$

$$\varepsilon_z^2 = \left| \mathbf{C}\boldsymbol{\sigma}_x\mathbf{C}^T + \mathbf{D}\boldsymbol{\sigma}_z\mathbf{D}^T \right|$$

Determinant of sum of 2×2 matrices…

$$|\mathbf{X} + \mathbf{Y}| = |\mathbf{X}| + |\mathbf{Y}| + tr\left\{\mathbf{X}^a\mathbf{Y}\right\},$$

where $\mathbf{X}^a$ is the adjoint of **X**…

$$\mathbf{X}^a = |\mathbf{X}|\mathbf{X}^{-1}, \quad |\mathbf{X}| \neq 0, \quad \text{or}$$

$$\mathbf{X}^a = \mathbf{J}^{-1}\mathbf{X}^T\mathbf{J}$$

and **J** is…

$$\mathbf{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{J}^2 = -\mathbf{I}.$$

The emittances are…

$$\varepsilon_x^2 = |\mathbf{A}|^2 \varepsilon_{x_0}^2 + |\mathbf{B}|^2 \varepsilon_{z_0}^2 + tr\left\{\left(\mathbf{A}\boldsymbol{\sigma}_x\mathbf{A}^T\right)^a \mathbf{B}\boldsymbol{\sigma}_z\mathbf{B}^T\right\},$$

$$\varepsilon_z^2 = |\mathbf{C}|^2 \varepsilon_{x_0}^2 + |\mathbf{D}|^2 \varepsilon_{z_0}^2 + tr\left\{\left(\mathbf{C}\boldsymbol{\sigma}_x\mathbf{C}^T\right)^a \mathbf{D}\boldsymbol{\sigma}_z\mathbf{D}^T\right\}$$

Now use alternate form for $\boldsymbol{\sigma}_{x,z}$ …

$$\boldsymbol{\sigma}_x = \varepsilon_{x_0}\mathbf{Q}_x\mathbf{Q}_x^T, \qquad \mathbf{Q}_x \equiv \frac{1}{\sqrt{\beta_x}}\begin{pmatrix} \beta_x & 0 \\ -\alpha_x & 1 \end{pmatrix},$$

and also use property of trace…

$$tr\{XYZ\} = tr\{YZX\} = tr\{ZXY\}$$

Emittances are then…

$$\varepsilon_x^2 = |\mathbf{A}|^2 \varepsilon_{x_0}^2 + |\mathbf{B}|^2 \varepsilon_{z_0}^2 + \varepsilon_{x_0}\varepsilon_{z_0} tr\left\{\mathbf{U}\mathbf{U}^T\right\},$$

$$\varepsilon_z^2 = |\mathbf{C}|^2 \varepsilon_{x_0}^2 + |\mathbf{D}|^2 \varepsilon_{z_0}^2 + \varepsilon_{x_0}\varepsilon_{z_0} tr\left\{\mathbf{V}\mathbf{V}^T\right\},$$

where

$$\mathbf{U} \equiv \mathbf{Q}_x^{-1}\mathbf{A}^a\mathbf{B}\mathbf{Q}_z,$$

$$\mathbf{V} \equiv \mathbf{Q}_x^{-1}\mathbf{C}^a\mathbf{D}\mathbf{Q}_z.$$

Now use the symplectic condition…

$$\mathbf{R}^T\mathbf{S}\mathbf{R} = \mathbf{R}\mathbf{S}\mathbf{R}^T = \mathbf{S} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}$$

$$\mathbf{R}^T\mathbf{S}\mathbf{R} = \begin{pmatrix} \mathbf{A}^T\mathbf{J}\mathbf{A} + \mathbf{C}^T\mathbf{J}\mathbf{C} & \mathbf{A}^T\mathbf{J}\mathbf{B} + \mathbf{C}^T\mathbf{J}\mathbf{D} \\ \mathbf{B}^T\mathbf{J}\mathbf{A} + \mathbf{D}^T\mathbf{J}\mathbf{C} & \mathbf{B}^T\mathbf{J}\mathbf{B} + \mathbf{D}^T\mathbf{J}\mathbf{D} \end{pmatrix} =$$

$$\mathbf{R}\mathbf{S}\mathbf{R}^T = \begin{pmatrix} \mathbf{A}\mathbf{J}\mathbf{A}^T + \mathbf{B}\mathbf{J}\mathbf{B}^T & \mathbf{A}\mathbf{J}\mathbf{C}^T + \mathbf{B}\mathbf{J}\mathbf{D}^T \\ \mathbf{C}\mathbf{J}\mathbf{A}^T + \mathbf{D}\mathbf{J}\mathbf{B}^T & \mathbf{C}\mathbf{J}\mathbf{C}^T + \mathbf{D}\mathbf{J}\mathbf{D}^T \end{pmatrix}$$

$$\left|\mathbf{A}^T\mathbf{J}\mathbf{A} + \mathbf{C}^T\mathbf{J}\mathbf{C}\right| = \left|\mathbf{A}\mathbf{J}\mathbf{A}^T + \mathbf{B}\mathbf{J}\mathbf{B}^T\right| = 1,$$

$$\left|\mathbf{B}^T\mathbf{J}\mathbf{B} + \mathbf{D}^T\mathbf{J}\mathbf{D}\right| = \left|\mathbf{C}\mathbf{J}\mathbf{C}^T + \mathbf{D}\mathbf{J}\mathbf{D}^T\right| = 1$$

and find the relations between the sub-matrix determinants…

$$|\mathbf{A}|+|\mathbf{C}|=\pm 1, \quad |\mathbf{A}|=|\mathbf{D}|, \quad |\mathbf{B}|=|\mathbf{C}|,$$

and also between **U** and **V**…

$$\mathbf{V}\equiv\mathbf{Q}_x^{-1}\mathbf{C}^a\mathbf{D}\mathbf{Q}_z=\mathbf{Q}_x^{-1}(\mathbf{J}^{-1}\mathbf{C}^T\mathbf{J})\mathbf{D}\mathbf{Q}_z,$$

and using from above…

$$\mathbf{C}^T\mathbf{J}\mathbf{D}=-\mathbf{A}^T\mathbf{J}\mathbf{B},$$

$$\mathbf{V}\equiv-\mathbf{Q}_x^{-1}\mathbf{J}^{-1}(\mathbf{A}^T\mathbf{J}\mathbf{B})\mathbf{Q}_z$$

$$=-\mathbf{Q}_x^{-1}\mathbf{A}^a\mathbf{B}\mathbf{Q}_z=-\mathbf{U}$$

Therefore,

$$tr\left\{\mathbf{U}\mathbf{U}^T\right\}=tr\left\{\mathbf{V}\mathbf{V}^T\right\}$$

which is the sum of squares of the normalized coupling matrix, and is positive.

$$tr\left\{\mathbf{U}\mathbf{U}^T\right\}=U_{11}^2+U_{12}^2+U_{21}^2+U_{22}^2\equiv\lambda^2\geq 0$$

The emittances are…

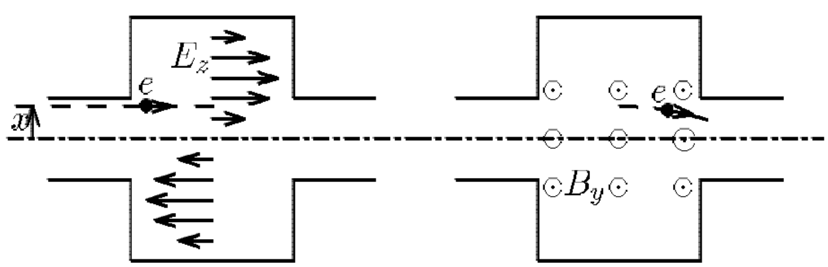
$$\varepsilon_x^2 = |\mathbf{A}|^2 \varepsilon_{x_0}^2 + (1 - |\mathbf{A}|)^2 \varepsilon_{z_0}^2 + \varepsilon_{x_0} \varepsilon_{z_0} \lambda^2,$$

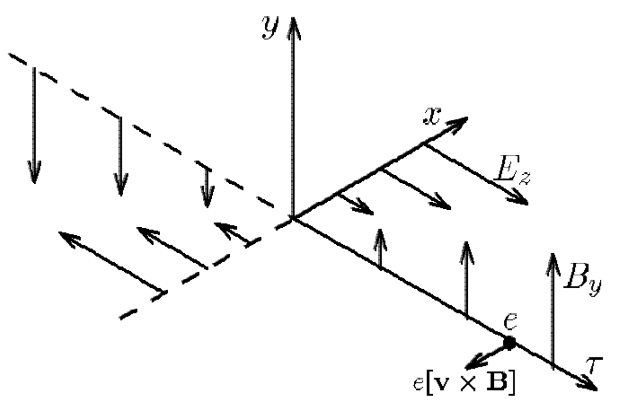
$$\varepsilon_z^2 = (1 - |\mathbf{A}|)^2 \varepsilon_{x_0}^2 + |\mathbf{A}|^2 \varepsilon_{z_0}^2 + \varepsilon_{x_0} \varepsilon_{z_0} \lambda^2.$$

Now introduce a rectangular transverse RF cavity operating in TM_{110} mode…

$$E_z = \frac{V_0}{\ell} \frac{x}{a} \cos \omega t \approx \frac{V_0}{\ell} \frac{x}{a},$$

$$B_y = \frac{V_0}{\ell} \frac{x}{a\omega} \sin \omega t \approx \frac{V_0}{\ell} \frac{z}{ca}.$$





Transverse RF accelerates and kicks beam…

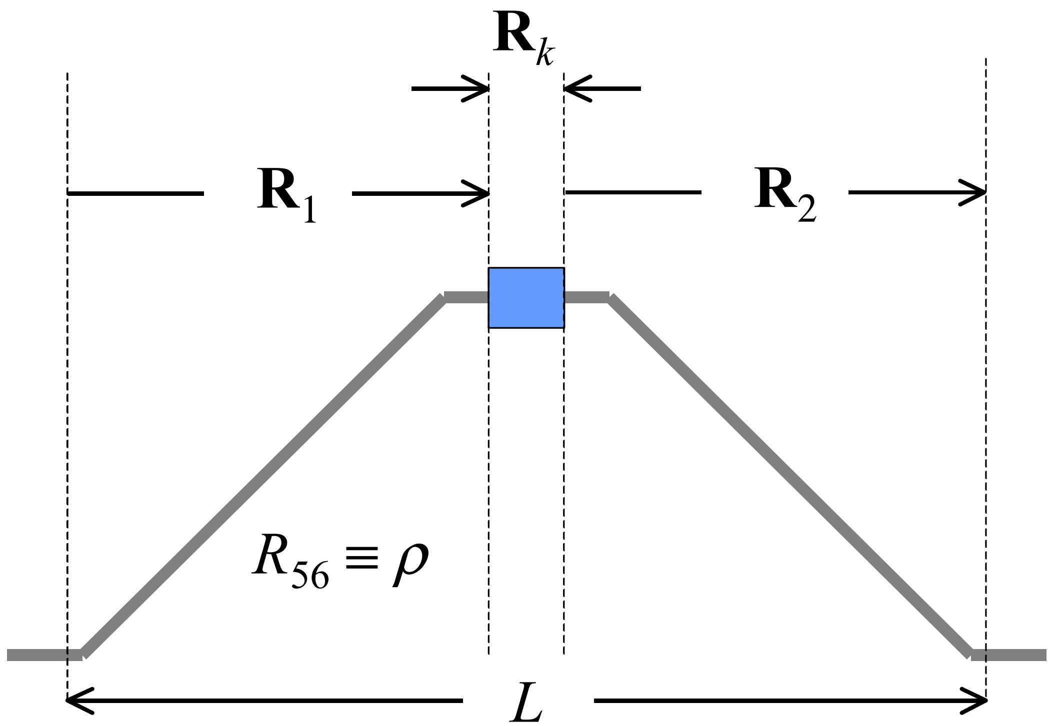
$$\delta \approx \frac{eV_0}{aE_0} x = kx, \qquad \Delta x' \approx \frac{eV_0}{aE_0} z = kz$$

The transfer matrix of this <u>thin-lens</u> cavity is

$$\mathbf{R}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix},$$

which *looks like a thin-lens skew quad*, but here we consider the space (x, x', z, δ).

Now place this cavity in a chicane…



$$\mathbf{R} = \mathbf{R}_2 \mathbf{R}_k \mathbf{R}_1$$

$$\mathbf{R}_1 = \begin{bmatrix} 1 & L/2 & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \rho/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_2 = \begin{bmatrix} 1 & L/2 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & -\eta & 1 & \rho/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}=\begin{bmatrix} 1+\eta k & L & kL/2 & k\left(\frac{\rho L}{4}-\eta^2\right) \\ 0 & 1-\eta k & k & k\rho/2 \\ k\rho/2 & k\left(\frac{\rho L}{4}-\eta^2\right) & 1+\eta k & \rho \\ k & kL/2 & 0 & 1-\eta k \end{bmatrix}$$

$$|\mathbf{A}|=|\mathbf{D}|=1-\eta^2 k^2, \quad |\mathbf{B}|=|\mathbf{C}|=\eta^2 k^2$$

Now work out λ^2…

$$\mathbf{U}=\mathbf{Q}_x^{-1}\mathbf{A}^a\mathbf{B}\mathbf{Q}_z^{-1}$$

$$\lambda^2=tr\left\{\mathbf{U}\mathbf{U}^T\right\}=4 \text{ terms, or for } k=-\frac{1}{\eta}...$$

$$\lambda^2=\frac{4\left(1+\alpha_x^2\right)\left(1+\alpha_z^2\right)}{k^2\beta_x\beta_z}=\frac{4\sigma_{x'}^2\sigma_\delta^2\eta^2}{\varepsilon_{x_0}\varepsilon_{z_0}}.$$

Emittance are…

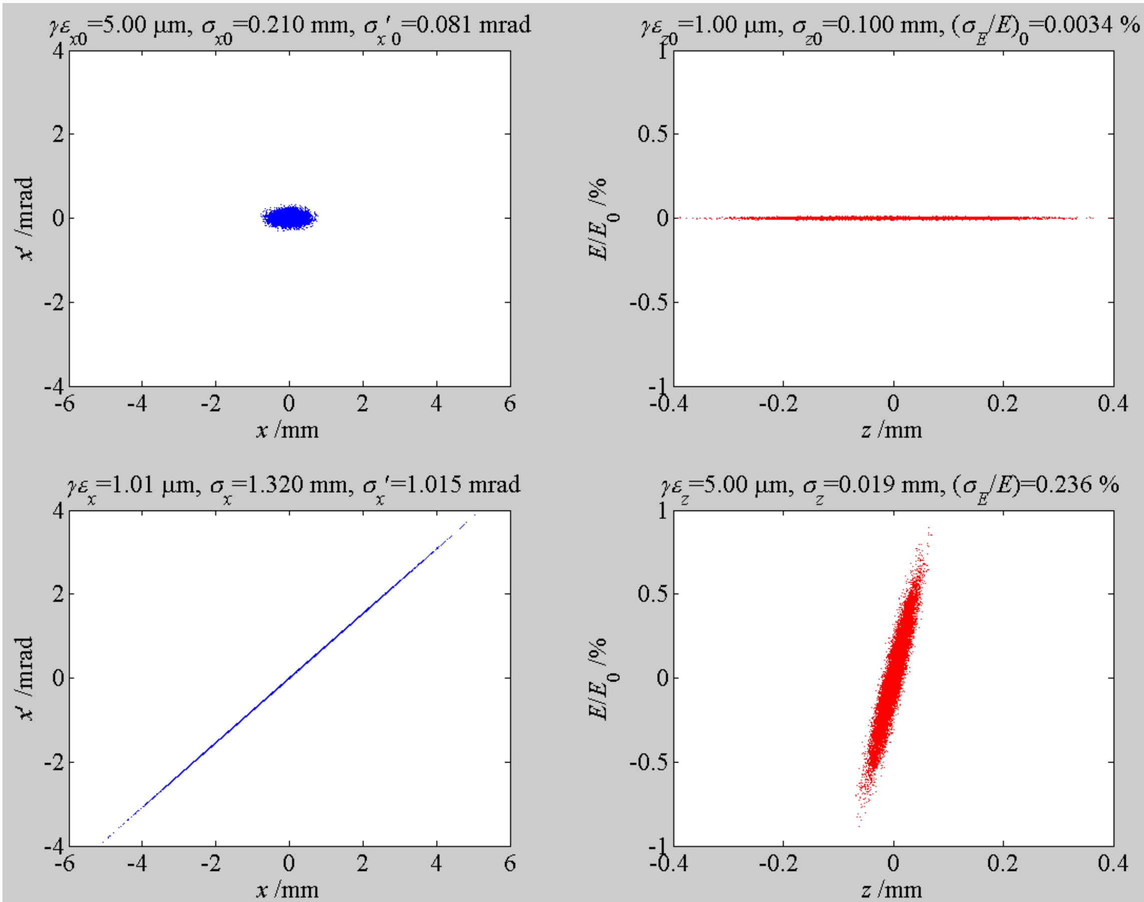
$$\varepsilon_x = \sqrt{\varepsilon_{z_0}^2 + 4\sigma_{x'}^2\sigma_\delta^2\eta^2} > \varepsilon_{z_0}$$
$$\varepsilon_z = \sqrt{\varepsilon_{x_0}^2 + 4\sigma_{x'}^2\sigma_\delta^2\eta^2} > \varepsilon_{x_0}$$

If $\varepsilon_{x0} = \varepsilon_{z0}$, then $\varepsilon_x = \varepsilon_z$ (i.e., always equal).

For example, take $\beta_x = 2.6$ m, $\sigma_z = 100$ μm, $\sigma_E = 5$ keV, $\beta_z = 2.9$ m, $\alpha_x = \alpha_z = 0$, $E_0 = 150$ MeV, $\eta = 100$ mm, $k = 10$ m^{-1}, $a = 1$ cm, $V_0 = 15$ MV…

$$\gamma\varepsilon_{x_0} = 5\ \mu\text{m} \rightarrow\ \ \gamma\varepsilon_x = \gamma\varepsilon_{z_0}\sqrt{1+0.026} \approx 1\ \mu\text{m}$$
$$\gamma\varepsilon_{z_0} = 1\ \mu\text{m} \rightarrow\ \ \gamma\varepsilon_z = \gamma\varepsilon_{x_0}\sqrt{1+0.001} \approx 5\ \mu\text{m}$$

Get nearly complete emittance exchange.

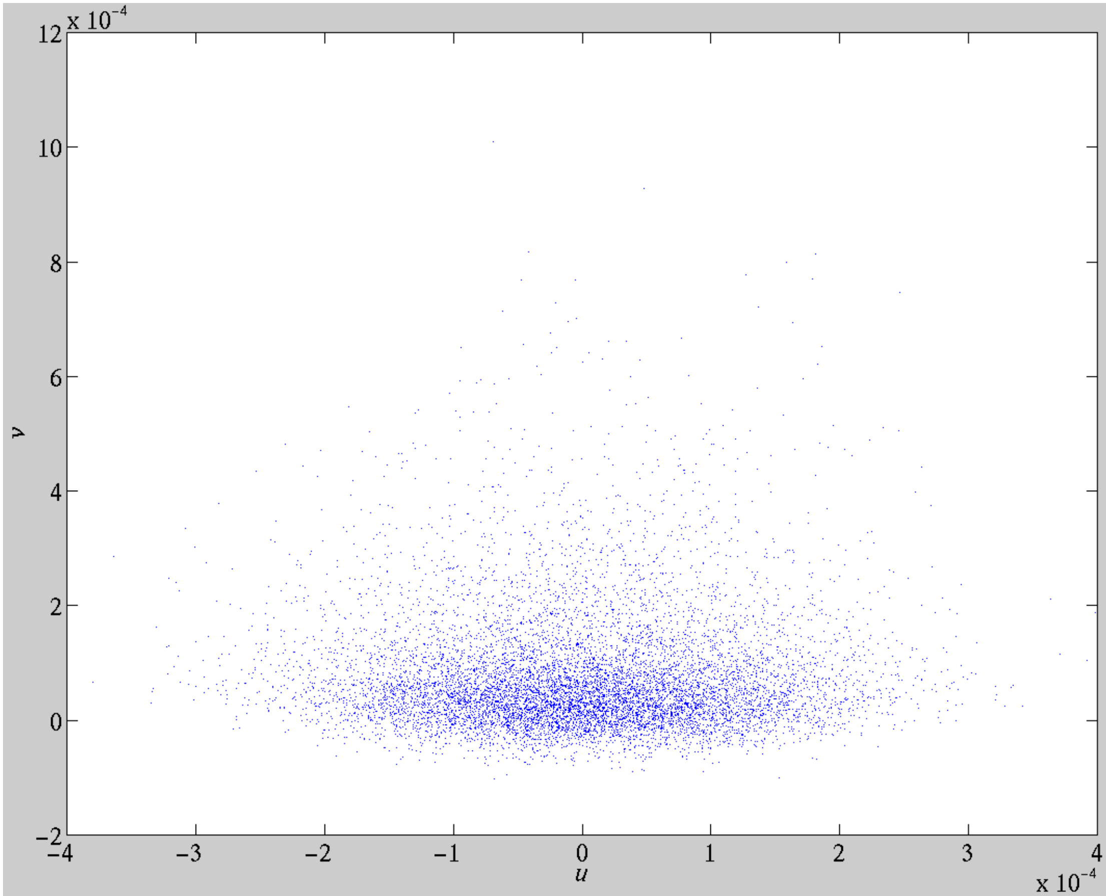


Bunch length for $k = -1/\eta$, and $\alpha_x = 0$ …

$$\sigma_z^2 = k^2 \varepsilon_{x_0} \left[\frac{\rho^2 \beta_x}{4} + \frac{1}{\beta_x} \left(\frac{\rho L}{4} - \frac{1}{k^2} \right)^2 \right] + \rho^2 \sigma_{\delta_0}^2$$

Bunch also typically <u>compresses</u>.

Second-order optics (T_{166}) can be significant, but use small β_x and/or large η (small k) for control…



➡ $\beta_x = 10$ m, $\eta = 50$ mm $\rightarrow \sigma_E/E \approx 0.8\%$

$\gamma\varepsilon_x \approx 3.1\ \mu$m (linear: 1 μm).
$\gamma\varepsilon_z \approx 10\ \mu$m (linear: 5 μm).

But, for $\beta_x = 2.6$ m, $\eta = 100$ mm $\rightarrow$ $\sigma_E/E \approx 0.2\%$ get…

$\gamma\varepsilon_x \approx 1.07\ \mu$m
$\gamma\varepsilon_z \approx 5.22\ \mu$m

Tolerances…

$1 \cdot \sigma$ centroid jitter becomes $1 \cdot \sigma$ jitter, but in the other plane:

$$\langle x_0 \rangle = \sigma_{x0} \quad \rightarrow \quad \langle z \rangle = \sigma_z, \ \langle \delta \rangle = \sigma_\delta$$
$$\langle x'_0 \rangle = \sigma_{x'0} \quad \rightarrow \quad \langle \delta \rangle \approx \sigma_\delta/2$$
$$\langle z_0 \rangle = \sigma_{z0} \quad \rightarrow \quad \langle x \rangle = \sigma_x, \ \langle x' \rangle = \sigma_{x'}$$
$$\langle \delta_0 \rangle = \sigma_{\delta 0} \quad \rightarrow \quad \langle z \rangle = \sigma_z/30$$

$$|\Delta V/V_0| < 0.5\% \quad \rightarrow \quad |\Delta\varepsilon/\varepsilon_x| < 3\ \%$$
$$|\Delta\phi| < 0.3^\circ \text{ S-band} \rightarrow \quad |\langle x \rangle| < \sigma_x, \ |\langle x' \rangle| < \sigma_{x'}$$

$|\alpha_z| < 2.5$ (i.e., projected energy spread up to 3-times larger than intrinsic spread, for these parameters)

$$\rightarrow \quad |\Delta\varepsilon/\varepsilon_x| < 10\ \%$$