CSR calculation by paraxial approximation

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Short Bunch

- Colliders — for high luminosity
- ERL — for short duration light
- FEL — for high peak current

Also high current may be required for their performance.

Future projects of KEK

- SuperKEKB (e^+e^- storage ring collider) $N \sim 1 \times 10^{11}$

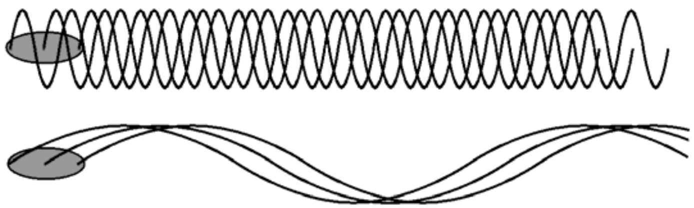
$$\sigma_z = 6\text{mm (KEKB)} \rightarrow 3\text{mm (SuperKEKB)}$$

- ERL (Energy Recovery Linac) $N \sim 5 \times 10^{8}$

$$\sigma_z = 1\text{ps} \sim 0.1\text{ps}$$

Coherent Synchrotron Radiation (CSR)

High frequency



Incoherent

Low frequency

Coherent

In storage rings – Bunch lengthening, Microwave instability, CSR burst

Topics

1. Introduction
2. Our approach to calculate CSR
3. Longitudinal instability due to CSR in SuperKEKB positron ring

CSR in storage rings

	Storage ring	ERL
Bunch length	1 ~ 10mm	0.01~ 0.1mm
Shielding by vacuum chamber	very strong	weak
State of CSR field in a bend	transient	transient ~ steady
Effects on bunch	emittance growth bunch lengthening beam instability	emittance growth beam instability

Shielding & Transient effect

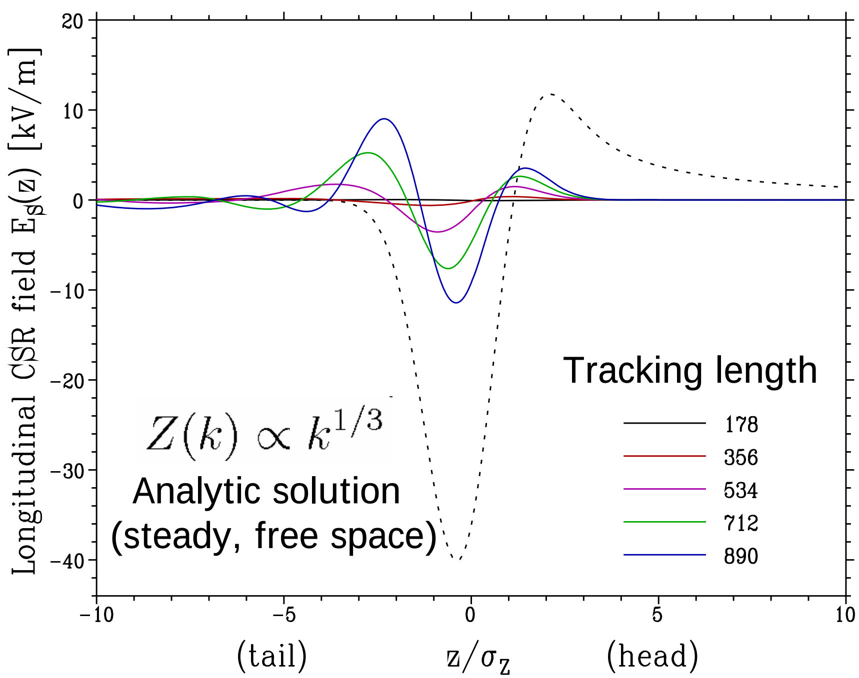
- Shielding effect : size of vacuum chamber = h

  shielding condition : $\lambda \gtrsim \sqrt{6h^3/\pi\rho}$

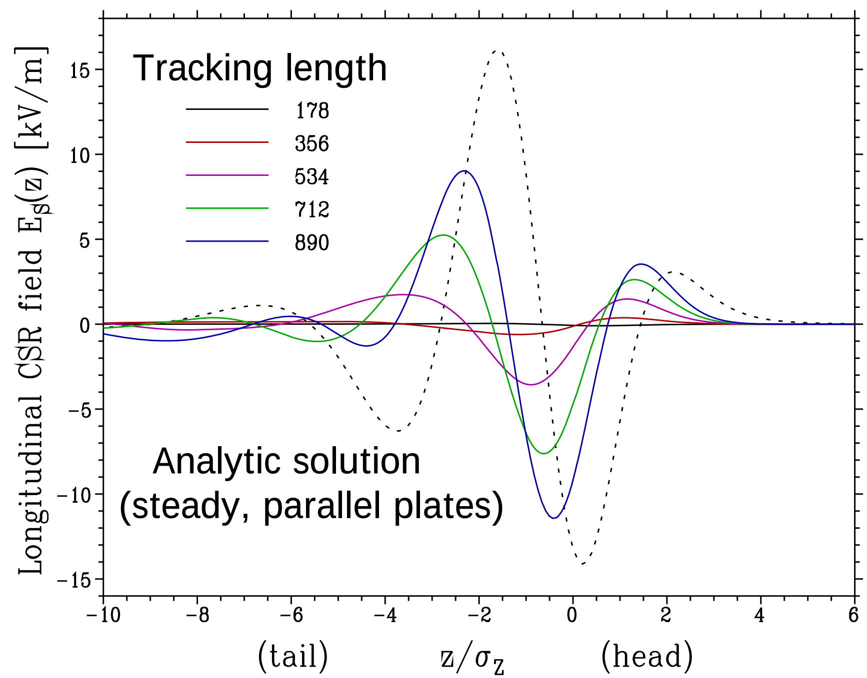
- Transient effect : length of bending magnet = L_m

  steady condition : $L_m{}^3/24\rho^2 \gg \sigma_s$

λ = wavelength
ρ = bending radius
σ_z = bunch length
$\sigma_z = 3\text{mm}$



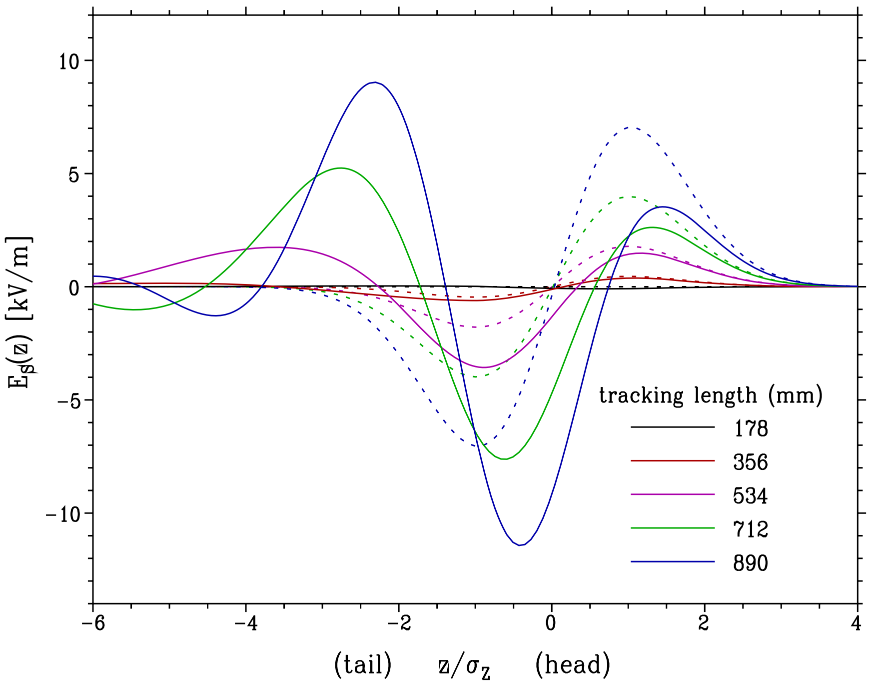
Neglect both shielding and transient effect



Consider only shielding effect
(= neglect transient effect)

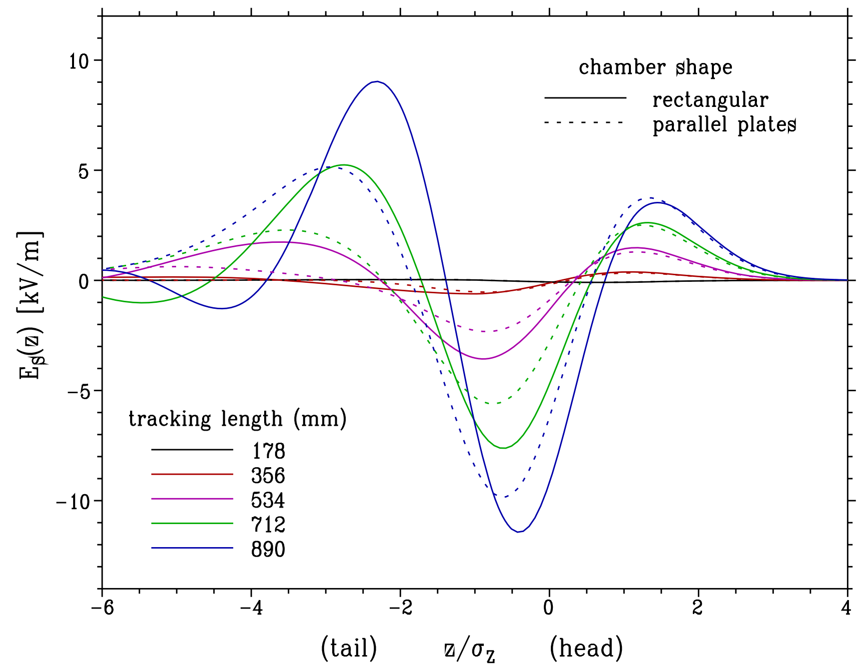
Consider only transient effect

(= neglect shielding effect)



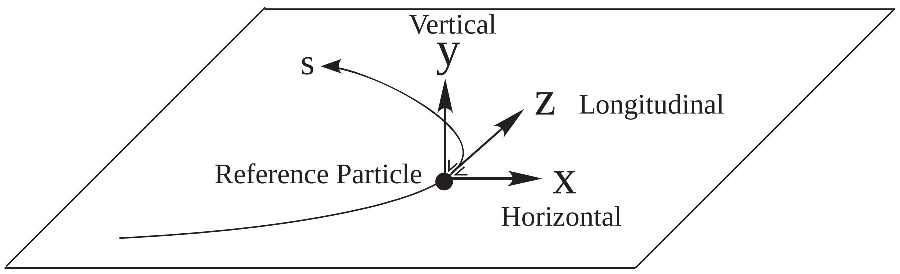
Consider pipe-shaped chamber

(= neglect side walls of chamber)



When we calculate CSR in a storage ring, we must consider both the vacuum pipe and the magnet length.

coordinate system



$$
\begin{aligned}
x &= \text{horizontal axis} \\
y &= \text{vertical axis} \\
z &= \text{longitudinal position} \quad = s - ct \\
s &= \text{independent variable}
\end{aligned}
$$

symbols

$$
\begin{aligned}
\rho &= \text{bending radius} \\
g &= 1 + \frac{x}{\rho} \quad \text{curvature factor} \\
k &= \text{wavenumber} \\
\check{\lambda}(z) &= \text{longitudinal charge distribution} \\
\check{J}_0(x, y, z) &= \text{charge distribution} \\
& \quad \text{e.g.} \quad \check{J}_0 = \check{\lambda}(z)\delta(x)\delta(y)
\end{aligned}
$$

$$
\begin{aligned}
\check{\boldsymbol{E}}(x, y, z; s) &= \text{Time domain variable} \\
\boldsymbol{E}(x, y, k; s) &= \text{Frequency domain variable}
\end{aligned}
$$

CSR calculation by paraxial approximation

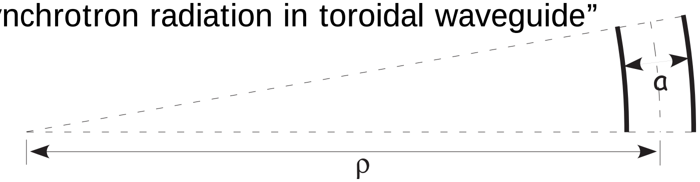
Mesh calculation of EM field (E,B) in a beam pipe

Assumptions

(a) Pipe size a is much smaller than the bending radius of the magnet.

$$\epsilon \equiv \sqrt{a/\rho} \ll 1$$

G.V.Stupakov, I.A.Kotelnikov, PRST-AB, 6, 034401 (2003)
"Shielding and synchrotron radiation in toroidal waveguide"

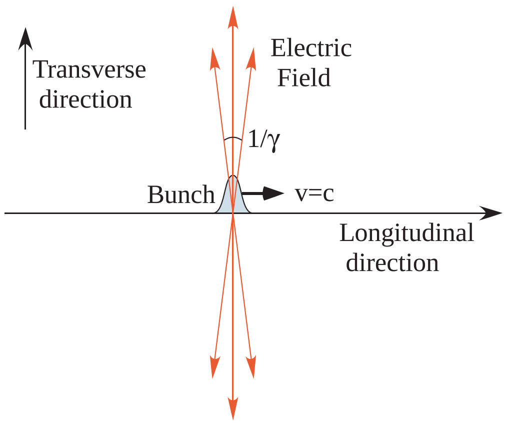


(b) Relativistic electrons : $\gamma \gg 1$

$$E_s \ll E_x, E_y$$

(c) Neglect backward waves (paraxial approximation)

Surface of the pipe must be smooth



(d) Bunch distribution does not change by CSR.

Predictable change can be considered.

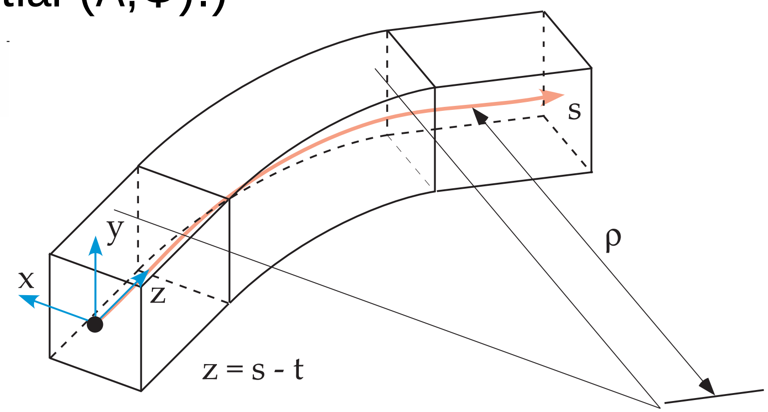
The dynamic variation of the bunch can be considered with particle tracking.

(1) Begin with Maxwell equations (E, B) in accelerator coordinates (x,y,z;s)

(We do not handle the retarded potential (A,Φ).)

Time domain : $\check{\boldsymbol{E}}(x,y,z;s),\ \check{\boldsymbol{B}}(x,y,z;s)$

$$z = s - ct$$



(2) Fourier transform EM field w.r.t z

$$\check{f}(s-ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f(k)\, e^{ik(s-ct)}$$

$$k = \text{wave number}$$

Frequency domain : $\boldsymbol{E}(x,y,k;s), \boldsymbol{B}(x,y,k;s)$

(3) Approximate these equations

Paraxial approximation

(4) Solve them by finite difference

Beam pipe = boundary condition

(5) Inverse Fourier transform
Back to the time domain

$$\check{\boldsymbol{E}}(x,y,z;s),\ \check{\boldsymbol{B}}(x,y,z;s)$$

Fourier transform

$$\text{Time domain}: \quad z, \ \check{f}(z)$$
$$\text{Frequency domain}: \quad k, \ f(k)$$

$$z \equiv s - t$$

- Definition

$$\check{f}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f(k)\, e^{ikz}$$
$$f(k) = \int_{-\infty}^{\infty} dz\, \check{f}(z)\, e^{-ikz}$$

Basis $e^{ikz} = e^{ik(s-ct)}$

Plane waves propagating forward at the speed of light

- Field evolution

$$\check{\boldsymbol{E}}(x,y,z;s),\ \check{\boldsymbol{B}}(x,y,z;s) \ \Rightarrow\ \check{f}(z,s)$$
$$\boldsymbol{E}(x,y,k;s),\ \boldsymbol{B}(x,y,k;s) \ \Rightarrow\ f(k,s)$$

$$s = \text{independent variable}$$

- Fourier transform of the derivatives

$$\frac{\partial \check{f}}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\,(-ik)\, f(k,s)$$
$$\frac{\partial^2 \check{f}}{\partial t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\,(-k^2)\, f(k,s)$$
$$\frac{\partial \check{f}}{\partial s} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\left(\frac{\partial}{\partial s} + ik\right) f(k,s)$$
$$\frac{\partial^2 \check{f}}{\partial s^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\left(\frac{\partial^2}{\partial s^2} + 2ik\frac{\partial}{\partial s} - k^2\right) f(k,s)$$

Differentiation with respect to s acts not only on the basis: exp(ik(s-t)) but also on the field: f(k,s) because we consider the field evolution.

Gauss's low: $\boldsymbol{\nabla} \cdot \check{\boldsymbol{E}} = \mu_0 \check{J}_0$

$$\frac{1}{g}\left(\frac{\partial(g\check{E}_x)}{\partial x} + \frac{\partial(g\check{E}_y)}{\partial y} + \frac{\partial \check{E}_s}{\partial s}\right) = \mu_0 \check{J}_0 \quad (1)$$

Fourier transform eq.(1) to the frequency domain, neglect small terms

$$E_s = \frac{i}{k}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} - \mu_0 J_0\right)$$

Magnetic field

$$B_x = -\left(1 - \frac{x}{\rho}\right)E_y + \frac{i}{k}\frac{\partial E_y}{\partial s} - \frac{i}{k}\frac{\partial E_s}{\partial y}$$

$$B_y = +\left(1 - \frac{x}{\rho}\right)E_x - \frac{i}{k}\frac{\partial E_x}{\partial s} + \frac{i}{k}\frac{\partial E_s}{\partial x}$$

$$B_s = \frac{i}{k}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

Lorentz force

$$F_x = +\frac{i}{2k}\left(\frac{\partial B_s}{\partial y} - \frac{\partial E_s}{\partial x}\right)$$

$$F_y = -\frac{i}{2k}\left(\frac{\partial B_s}{\partial x} + \frac{\partial E_s}{\partial y}\right)$$

All field components are given by the transverse E-field: E_x and E_y.

Fundamental equation

From Maxwell equations,

$$\nabla(\nabla\cdot\check{\boldsymbol{E}})-\nabla\times(\nabla\times\check{\boldsymbol{E}})-\frac{\partial^2\check{\boldsymbol{E}}}{\partial t^2}=\mu_0\left(\nabla\check{J}_0+\frac{\partial\check{\boldsymbol{J}}}{\partial t}\right) \qquad (2)$$

$$\begin{aligned}\nabla(\nabla\cdot\check{\boldsymbol{E}})-\nabla\times(\nabla\times\check{\boldsymbol{E}})=&\\ &\left[\frac{\partial}{\partial x}\left(\frac{1}{g}\frac{\partial(g\check{E}_x)}{\partial x}\right)+\frac{\partial^2\check{E}_x}{\partial y^2}+\frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial\check{E}_x}{\partial s}\right)+\hat{\mathcal{L}}\check{E}_s\right]\boldsymbol{e}_x\\ &+\left[\frac{1}{g}\frac{\partial}{\partial x}\left(g\frac{\partial\check{E}_y}{\partial x}\right)+\frac{\partial^2\check{E}_y}{\partial y^2}+\frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial\check{E}_y}{\partial s}\right)\right]\boldsymbol{e}_y\\ &+\left[\frac{\partial}{\partial x}\left(\frac{1}{g}\frac{\partial(g\check{E}_s)}{\partial x}\right)+\frac{\partial^2\check{E}_s}{\partial y^2}+\frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial\check{E}_s}{\partial s}\right)-\hat{\mathcal{L}}\check{E}_x\right]\boldsymbol{e}_s\end{aligned} \qquad (3)$$

← Horizontal direction (the $\boldsymbol{e}_x$ term)

← Vertical direction (the $\boldsymbol{e}_y$ term)

where $$\hat{\mathcal{L}}\check{E}_\alpha=2\frac{\partial\check{E}_\alpha}{\partial s}\frac{\partial}{\partial x}\frac{1}{g}-\frac{\check{E}_\alpha}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial g}{\partial x}\right) \qquad (4)$$

Fourier transform of Eq.(2) is given by

$$\begin{aligned}&\frac{\partial^2E_x}{\partial x^2}+\frac{1}{g\rho}\frac{\partial E_x}{\partial x}-\frac{E_x}{g^2\rho^2}+\frac{\partial^2E_x}{\partial y^2}\\ &+\frac{1}{g^2}\left(\frac{\partial^2E_x}{\partial s^2}+2ik\frac{\partial E_x}{\partial s}-k^2E_x\right)+k^2E_x-\frac{2}{g^2\rho}\left(\frac{\partial E_s}{\partial s}+ikE_s\right)\\ &-\frac{x}{g^3}\left(\frac{\partial E_x}{\partial s}+ikE_x\right)\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)-\frac{E_s}{g}\left(1-\frac{x}{g\rho}\right)\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)=\mu_0\frac{\partial J_0}{\partial x}\end{aligned} \qquad (5)$$

Neglect higher order terms $O(\epsilon^2)$

$$\left(\frac{\partial^2}{\partial s^2} + 2ik\frac{\partial}{\partial s} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2k^2 x}{\rho}\right) E_x = \mu_0 \frac{\partial J_0}{\partial x} + C_x$$
$$\left(\frac{\partial^2}{\partial s^2} + 2ik\frac{\partial}{\partial s} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2k^2 x}{\rho}\right) E_y = \mu_0 \frac{\partial J_0}{\partial y} + C_y \quad (6)$$

where

$$C_x = x\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)\left[ikE_x + \frac{\partial E_x}{\partial s}\right] + \left(\frac{\partial}{\partial s}\frac{1}{\rho}\right) E_s \quad (7)$$

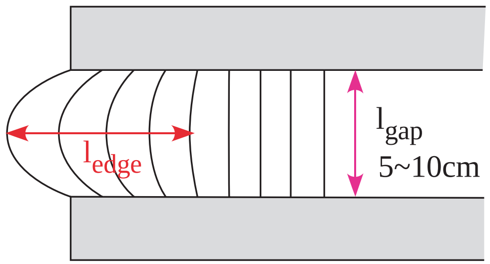
$$C_y = x\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)\left[ikE_y + \frac{\partial E_y}{\partial s}\right] \quad (8)$$

C_x and C_y come from the change of curvature at the edge of bending magnet.

Compare first term in C_x, C_y with second term in eq.(6)

$$\int_{s_1}^{s_2} 2ik\frac{\partial E_{x,y}}{\partial s} ds \sim \int_{s_1}^{s_2} ikxE_{x,y}\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right) ds$$
$$\delta E_{x,y} \sim \pm\frac{x}{2\rho} E_{x,y} = O(\epsilon^2)$$

small



Assuming that s-dependence of the field is weak,
neglect the term of 2nd derivative with respect to s:

$$\frac{\partial^2 \boldsymbol{E}_\perp}{\partial s^2} \ll 2ik\frac{\partial \boldsymbol{E}_\perp}{\partial s}$$

Equation to describe CSR

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s} = \frac{i}{2k}\left[\left(\boldsymbol{\nabla}_\perp^2 + \frac{2k^2 x}{\rho}\right)\boldsymbol{E}_\perp - \mu_0 \boldsymbol{\nabla}_\perp J_0\right]$$

Equation of Evolution

- First derivative with respect to s

  Field evolution (transient behavior) along the beam line

  We can solve it numerically step by step with respect to s.

- Ex and Ey are decoupled.

  If the boundary is a rectangular pipe, i.e., chamber walls are always parallel or perpendicular to the orbit plane, Ex and Ey can be independently calculated.

Equation of evolution

Usually, mesh size must be $\Delta x \ll \lambda / 2\pi$ in EM field analysis.

Our method ignores 2nd derivative,

backward waves $e^{ik(s+t)}$ are ignored.

The field consists of only forward waves.

$$e^{ik(s-t)}$$

We can factor the plane waves out of the EM field via Fourier transform.

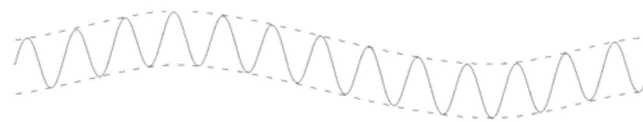
$$\check{\boldsymbol{E}}(x, y, z; s) \propto \boldsymbol{E}(x, y, k; s)\, e^{ik(s-t)}$$

We handle only $\boldsymbol{E}(x, y, k; s)$ which slowly changes along the beam line.

Mesh size can be larger than the actual field wavelength.

$e^{ik(s-t)}$:sinusoidal wave

$\check{E}(\boldsymbol{x}; t)$:actual wave



$\Downarrow$ Fourier Transform

$E(\boldsymbol{x}; k)$

slow change

The term of 1st derivative w.r.t. s describes the evolution of the field.

What is paraxial approximation ?

Originally, the paraxial approximation is a technique for LASER analysis.

Consider a laser beam propagating in a crystal whose index of refraction is not uniform.

Laser beam has strong directivity also in the crystal, however, laser is no longer the plane wave in it.

From Maxwell equations in the cristal with Cartesian coordinates,

$$\varepsilon\mu \frac{\partial^2 \check{\boldsymbol{E}}}{\partial t^2} = \boldsymbol{\nabla}^2 \check{\boldsymbol{E}} \qquad (3)$$

Laser is not a plane wave in the crystal but still similar to plane wave.

$$\check{\boldsymbol{E}}(\boldsymbol{r}, t) = \boldsymbol{E}(x, y, z)\, e^{i(kz-\omega t)}$$

Paraxial ray

A ray propagating almost parallel to the optical axis

Eq.(3) becomes

$$-2ik\frac{\partial \boldsymbol{E}}{\partial z} = \boldsymbol{\nabla}_{\perp}^2 \boldsymbol{E} + (\varepsilon\mu\omega^2 - k^2)\boldsymbol{E} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2}$$

Neglect the term of second derivative with respect to z,

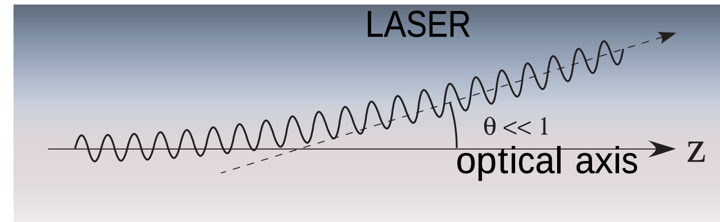
- Equation of laser in a crystal

$$\frac{\partial \boldsymbol{E}}{\partial z} = \frac{i}{2k}\left(\boldsymbol{\nabla}_\perp^2 + \left(n^2 - 1\right) k^2\right)\boldsymbol{E} \qquad (4)$$

n = index of refraction of the crystal

$$n(\boldsymbol{x}) = c\sqrt{\varepsilon\mu}$$

Laser is bent because of the non-uniform medium.



(in crystal)

- Equation of evolution without source term

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s} = \frac{i}{2k}\left(\boldsymbol{\nabla}_\perp^2 + \frac{2x}{\rho}k^2\right)\boldsymbol{E}_\perp \qquad (5)$$

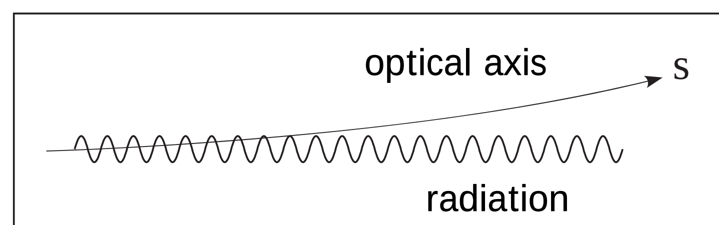
Index of refraction of the bending magnet

$$n = \sqrt{1 + \frac{2x}{\rho}} \approx 1 + \frac{x}{\rho} = g$$

Eq.(5) says that light is bent in vacuum.

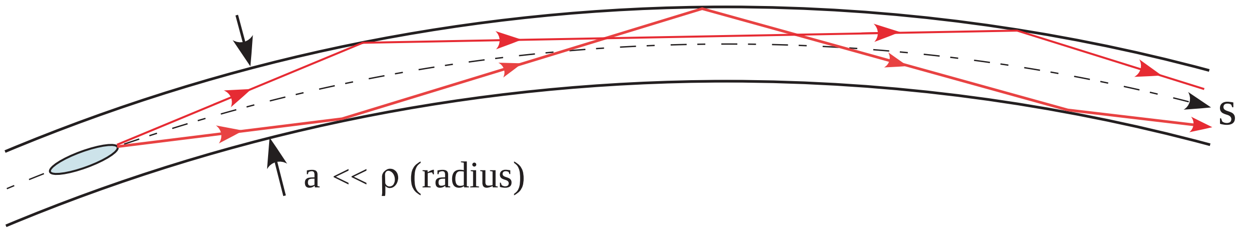
Our optical axis is curved.

(in bending magnet)



Role of beam pipe

Beam pipe is necessary in our approach.



The light, emitted from a bunch, cannot deviate from the s-axis due to the reflection on the pipe wall.

The radiation always propagates near around the axis.

The assumption $\sqrt{a/\rho} \ll 1$ is the condition so that the radiation field can be a paraxial ray.

Paraxial approximation works because of the beam pipe.

Schrödinger equation

Klein-Gordon equation (m = rest mass)

$$\nabla^2\Psi - \frac{\partial^2\Psi}{\partial t^2} = \frac{m^2}{\hbar^2}\Psi$$

In the nonrelativistic limit: $m \gg p$

$$E = \sqrt{m^2 + p^2} \quad \Rightarrow \quad m + p^2/2m$$
$$e^{-i\omega t} = e^{-i(E/\hbar)t} \quad \Rightarrow \quad e^{-i(m/\hbar)t} e^{-i(p^2/2m\hbar)t}$$

Factor the plane wave $e^{-i(m/\hbar)t}$ out of the wave function, deal only with the rest part

$$\Psi(\boldsymbol{x}, t) = \psi(\boldsymbol{x}, t) e^{-i(m/\hbar)t}$$

$$\nabla^2\psi - \frac{\partial^2\psi}{\partial t^2} + i\frac{2m}{\hbar}\frac{\partial\psi}{\partial t} = 0$$

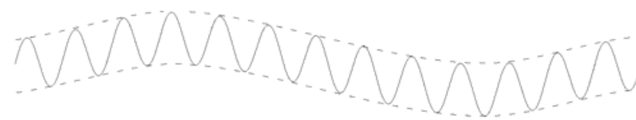
Neglect the term of 2nd derivative w.r.t. time,

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\nabla^2 - V\right]\psi$$

— Schrödinger equation —

$e^{-i(m/\hbar)t}$:sinusoidal wave

Ψ :actual wave function



$\Downarrow$ factor $e^{-i(m/\hbar)t}$ out

ψ: rest part

slow change

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s} = \frac{i}{2k}\left(\nabla_\perp^2 + \frac{2x}{\rho}k^2\right)\boldsymbol{E}_\perp$$

Equation of evolution without source term

Scale length of Field

Equation of evolution without source term

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s} = \frac{i}{2k}\left(\boldsymbol{\nabla}_\perp^2 + \frac{2k^2}{\rho}x\right)\boldsymbol{E}_\perp$$

Normalize x, y, s with dimensionless variables

$$x = \ell_x \hat{x}$$
$$y = \ell_y \hat{y}$$
$$s = \ell_s \hat{s}$$

The equation becomes

$$-i\frac{\partial \boldsymbol{E}_\perp}{\partial \hat{s}} = 2\left\{\frac{\ell_s}{k\ell_x^2}\right\}\frac{\partial \boldsymbol{E}_\perp}{\partial \hat{x}} + 2\left\{\frac{\ell_s}{k\ell_y^2}\right\}\frac{\partial \boldsymbol{E}_\perp}{\partial \hat{y}} + \left\{\ell_s \ell_x \frac{k}{\rho}\right\}\hat{x}\boldsymbol{E}_\perp$$

put { }=1

Typical scale length of the field

transverse $\ell_x, \ell_y \sim \left(\rho/k^2\right)^{1/3}$

longitudinal $\ell_s \sim \left(\rho^2/k\right)^{1/3}$

⟷

$\lambda \gtrsim \sqrt{6h^3/\pi\rho}$

$L_m^3/24\rho^2 \gg \sigma_s$

Mesh size to resolve the field

transverse $\Delta x, \Delta y \ll (\rho/k^2)^{1/3}$

longitudinal $\Delta s \ll (\rho^2/k)^{1/3}$

1/5 ~ 1/10 is enough.

Examples to which this approach cannot be applied

- Free space or very large vacuum chamber
  - EM field is no longer a paraxial ray.
- Chamber structure so that backward waves are produced
  - Bellows, Cavity ➔ Chamber wall must be smooth.
- Ultra-short bunch, or fine structure in the bunch
  - Fine mesh is required to resolve the field. (expensive)

    The shortest bunch length I computed is 10 microns in 6cm pipe.
  - Bunch profile with sharp edge, e.g. rectangular, triangular, etc

    ➔ Bunch profile must be smooth.

Flexibility of this approach

- Bending radius does not have to be a constant but can be a function of s.

  Varying the radius ➔ Arbitrary smooth beam line can be simulated.

  - One can consider fringe field of magnet if needed.
  - Calculation can be performed also in the drift space.
  - CSR in wigglers

$$K(s) = 1/\rho(s) = K_0 \sin(k_w s)$$
$$K(s) = 0 \rightarrow \rho_0 \rightarrow 0 \rightarrow -\rho_0 \rightarrow 0$$

- Chamber cross section does not have to be uniform along the beam line if the chamber does not produce backward waves.

Consider a vacuum chamber whose cross section gradually varies along the beam line, one can obtain the EM field.

➔ Collimator impedance

- Predictable change of bunch profile such as bunch compressor
- Electrons of a finite energy

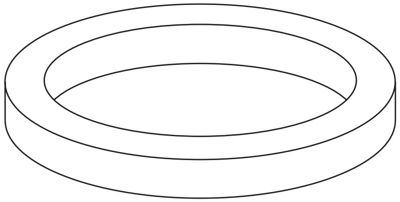
G.V.Stupakov, I.A.Kotelnikov, PRST-AB, 6, 034401 (2003)

"Shielding and synchrotron radiation in toroidal waveguide"

$$\left(\nabla_\perp^2 + 2k^2\left\{\frac{x}{\rho} - \frac{1}{2\gamma^2}\right\}\right)\boldsymbol{E}_\perp = 0$$

$$\check{f}(s - vt) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f(k)\, e^{ik(s-vt)}$$

Eigenvalue problem



Spectrum is discrete because of the eigenmodes.

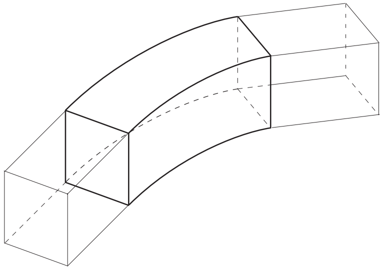
T.Agoh, K.Yokoya, PRST-AB, 7, 054403 (2004)

Equation of evolution

$$\left(\nabla_\perp^2 + 2k^2\frac{x}{\rho}\right)\boldsymbol{E}_\perp = -2ik\frac{\partial \boldsymbol{E}_\perp}{\partial s} + \mu_0\nabla_\perp J_0$$

$$\check{f}(s - ct, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f(k, s)\, e^{ik(s-ct)}$$

Initial value problem



Continuous spectrum

Maxwell equations with a finite energy in the frequency domain

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{1}{g\rho}\frac{\partial E_x}{\partial x}-\frac{E_x}{g^2\rho^2}+\frac{\partial^2 E_x}{\partial y^2}$$
$$+\frac{1}{g^2}\left(\frac{\partial^2 E_x}{\partial s^2}+2ik\frac{\partial E_x}{\partial s}-k^2E_x\right)+v^2k^2E_x-\frac{2}{g^2\rho}\left(\frac{\partial E_s}{\partial s}+ikE_s\right)$$
$$-\frac{x}{g^3}\left(\frac{\partial E_x}{\partial s}+ikE_x\right)\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)-\frac{E_s}{g}\left(1-\frac{x}{g\rho}\right)\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)=\mu_0\frac{\partial J_0}{\partial x}$$

Ignoring small terms,

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s}=\frac{i}{2k}\left[\left\{\nabla_\perp^2+2k^2\left(\frac{x}{\rho}-\frac{1}{2\gamma^2}\right)\right\}\boldsymbol{E}_\perp-\mu_0\nabla_\perp J_0\right]$$

which has an error: $\Delta=\frac{2ik}{\rho}E_s\Big/2k^2\frac{x}{\rho}E_x\sim\frac{1}{ka}\frac{E_s}{E_x}\sim\frac{\sigma_z}{a}\frac{1}{\gamma}$

(e.g.)

bunch length	$\sigma_z=0.3\,\mathrm{mm}$	
chamber radius	$a=30\,\mathrm{mm}$	
energy	$\gamma=10\ (5\,\mathrm{MeV})$	

➔ Relative error $\Delta\sim10^{-3}$

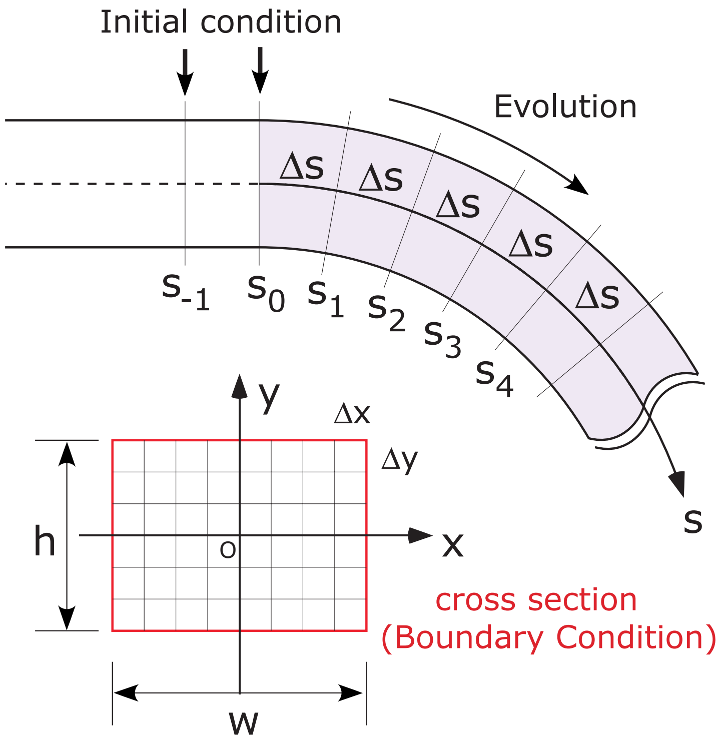
Algorithm

Solve equation of evolution with boundary condition

$$\frac{\partial \boldsymbol{E}_\perp}{\partial s} = \frac{i}{2k}\left[\left(\boldsymbol{\nabla}_\perp^2 + \frac{2k^2 x}{\rho}\right)\boldsymbol{E}_\perp - \mu_0 \boldsymbol{\nabla}_\perp J_0\right]$$

Discretize the equation by central difference: $s = \ell \Delta s$

$$\frac{\partial E}{\partial s} = \frac{E_{\ell+1} - E_{\ell-1}}{2\Delta s}$$



Solve initial condition at the entrance of bending magnet (radius=∞)

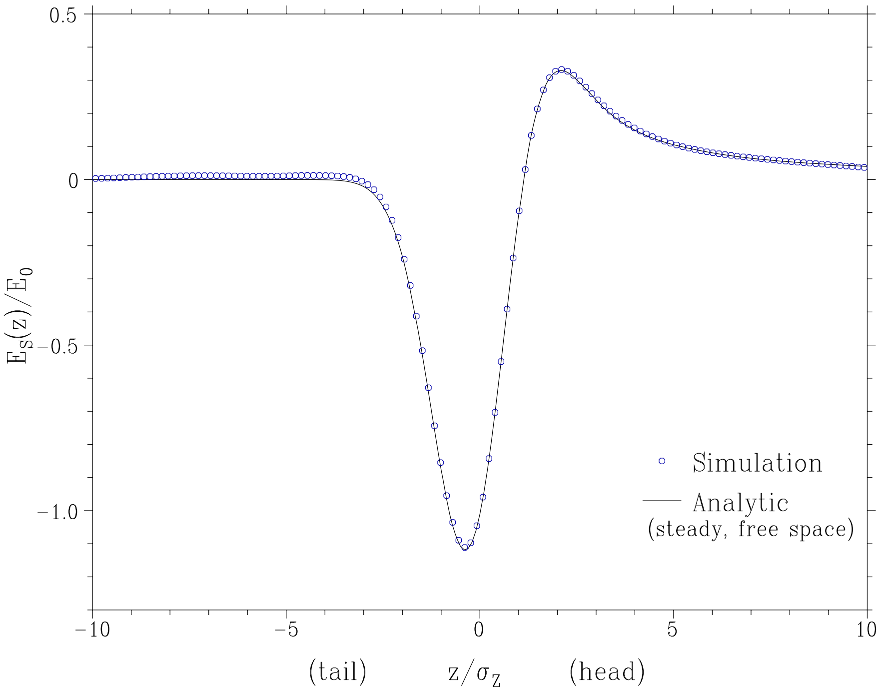
$$\boldsymbol{\nabla}_\perp^2 \boldsymbol{E}_\perp = \mu_0 \boldsymbol{\nabla}_\perp J_0$$

Proceed field evolution step by step along s-axis

$$E_{\ell+1} = E_{\ell-1} + 2\Delta s \left(\frac{\partial E}{\partial s}\right)$$

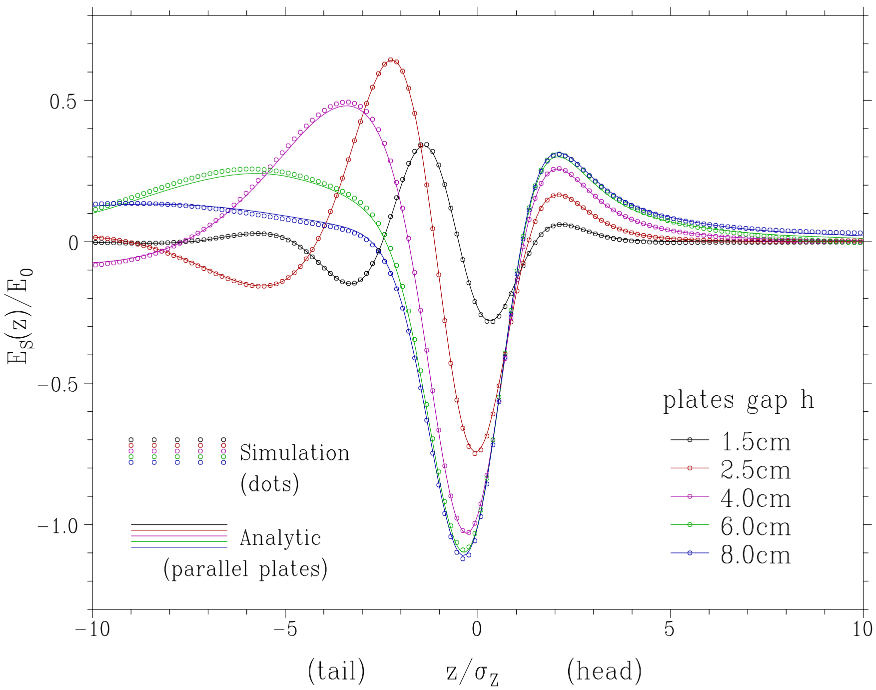
Comparison of steady CSR

Longitudinal E-field in free space



chamber size: w=34cm, h=28cm

Longitudinal E-field between parallel plates



chamber width: w=50cm

CSR in a steady state

Equation of evolution in a steady state

$$\left(\boldsymbol{\nabla}_\perp^2 + \frac{2k^2}{\rho}x\right)\boldsymbol{E}_\perp = \mu_0 \boldsymbol{\nabla}_\perp J_0$$

Assuming free space, the exact solution can be obtained analytically

$$\begin{aligned} E_s(0) &= -\frac{\mu_0 \kappa^2 \lambda}{2k}\int_{-\infty}^{\infty} dk_y \Big[\mathrm{Ai}'(k_y^2)\mathrm{Ci}'(k_y^2) + k_y^2\,\mathrm{Ai}(k_y^2)\mathrm{Ci}(k_y^2)\Big] \\ &= -\mu_0\lambda\left(\frac{4k}{\rho^2}\right)^{1/3}\int_0^{\infty} dk_y \Big[\{\mathrm{Ai}'(k_y^2)\}^2 + k_y^2\{\mathrm{Ai}(k_y^2)\}^2 \\ &\qquad - i\{\mathrm{Ai}'(k_y^2)\mathrm{Bi}'(k_y^2) + k_y^2\,\mathrm{Ai}(k_y^2)\mathrm{Bi}(k_y^2)\}\Big] \\ &= -\frac{\mu_0\lambda}{2\pi}\left(\frac{k}{3\rho^2}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)e^{i\pi/6} \end{aligned}$$

$$\mathrm{Ci}(\zeta) \equiv \mathrm{Ai}(\zeta) - i\mathrm{Bi}(\zeta)$$

Considering infinite parallel plates, we can solve it.

$$Z(\omega) = Z_0\frac{2\pi}{h}\left(\frac{2c}{\omega\rho}\right)^{1/3}\sum_{p=0}^{\infty}\Big\{\mathrm{Ai}'(\beta_p^2)\Big(\mathrm{Ai}'(\beta_p^2) - i\mathrm{Bi}'(\beta_p^2)\Big) + \beta_p^2\mathrm{Ai}(\beta_p^2)\Big(\mathrm{Ai}(\beta_p^2) - i\mathrm{Bi}(\beta_p^2)\Big)\Big\}$$

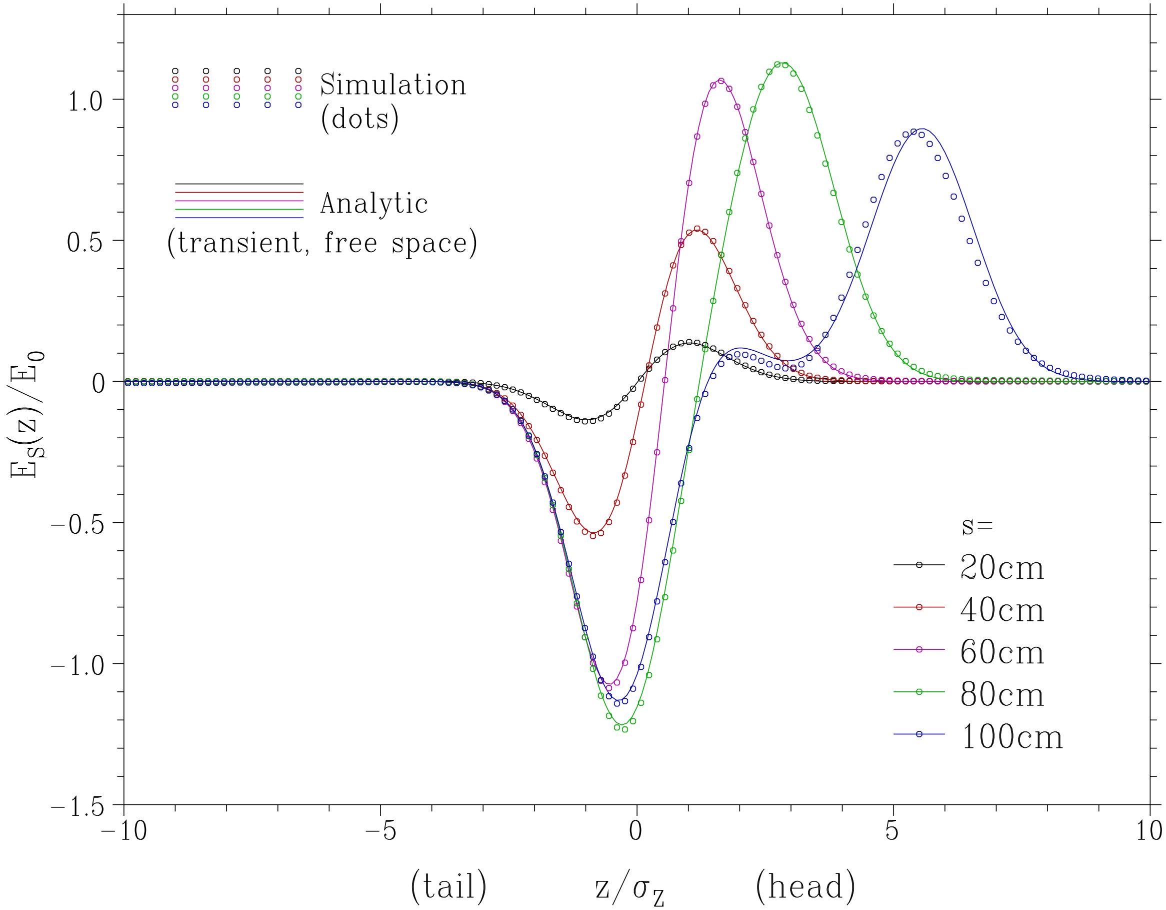
Also this impedance can be obtained by taking a limit in equation:

$$\frac{\bar{Z}(n,\omega)}{n} = \frac{2\pi^2 Z_0}{\beta}\frac{\rho}{h}\sum_{p=0}^{\infty}\Lambda_p\left[\frac{\beta\omega\rho}{nc}J_n'(\gamma_p\rho)H_n'^{(1)}(\gamma_p\rho) + \frac{\alpha_p^2}{\gamma_p^2}J_n(\gamma_p\rho)H_n^{(1)}(\gamma_p\rho)\right]$$

n → infinity $n\lambda_n = 2\pi\rho$ circular motion

R. Warnock, SLAC-PUB-5375 (1990)

Transient CSR in free space

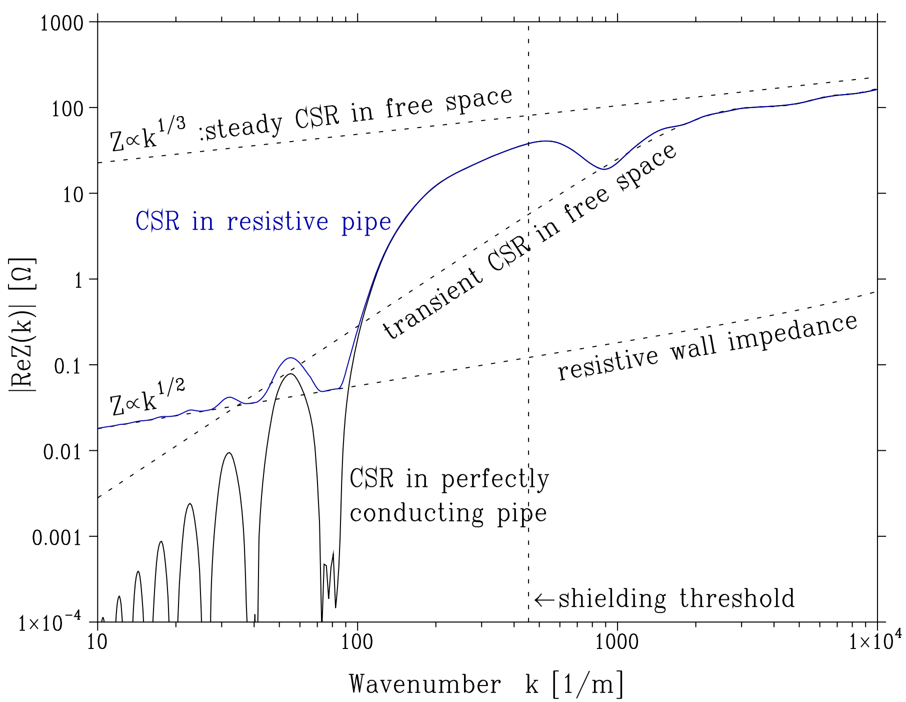


$$\frac{dE}{ds} = -\frac{2Ne^2}{4\pi\varepsilon_0(3\rho^2)^{1/3}}\left[\frac{\lambda(z-\zeta)-\lambda(z-4\zeta)}{\zeta^{1/3}} + \int_{z-\zeta}^{z}\frac{dz'}{(z-z')^{1/3}}\frac{d\lambda(z')}{dz'}\right] \qquad \zeta = \frac{s^3}{24\rho^2}$$

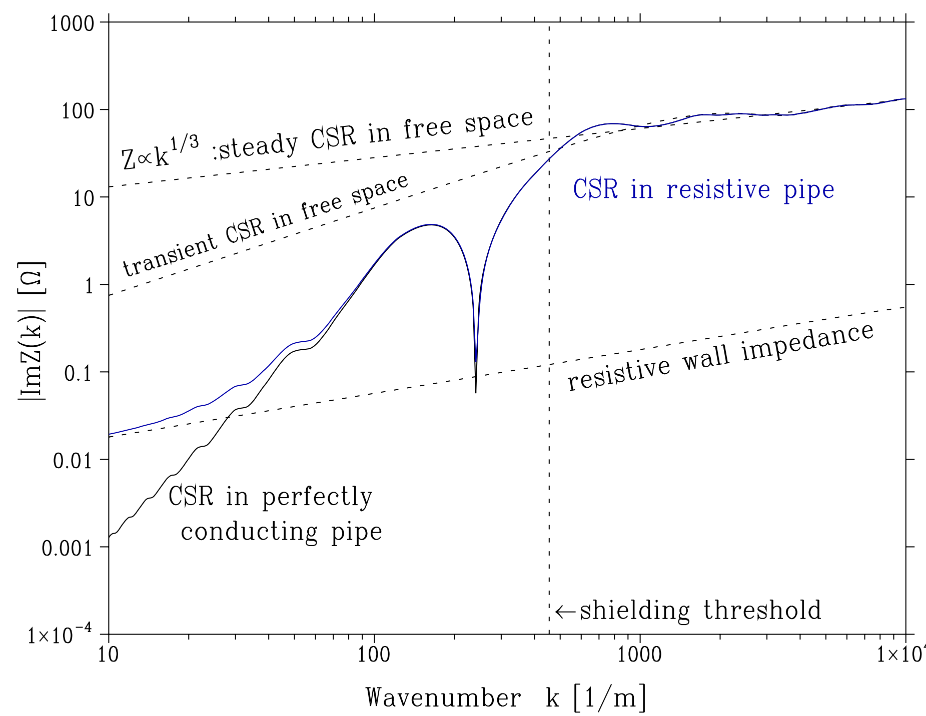
E.L.Saldin, E.A.Schneidmiller, M.V.Yurkov, Nucl.Inst.Meth. A398, p373 (1997)

Longitudinal impedance in a copper pipe (10cm square, R=10m, L_{mag}=1m)

Real part



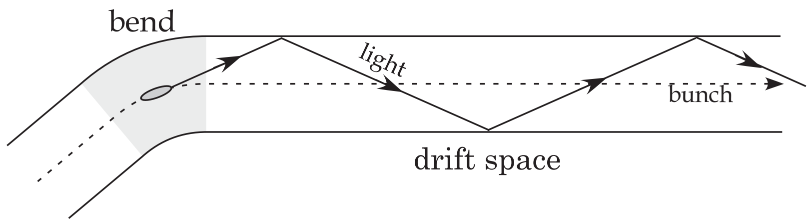
Imaginary part



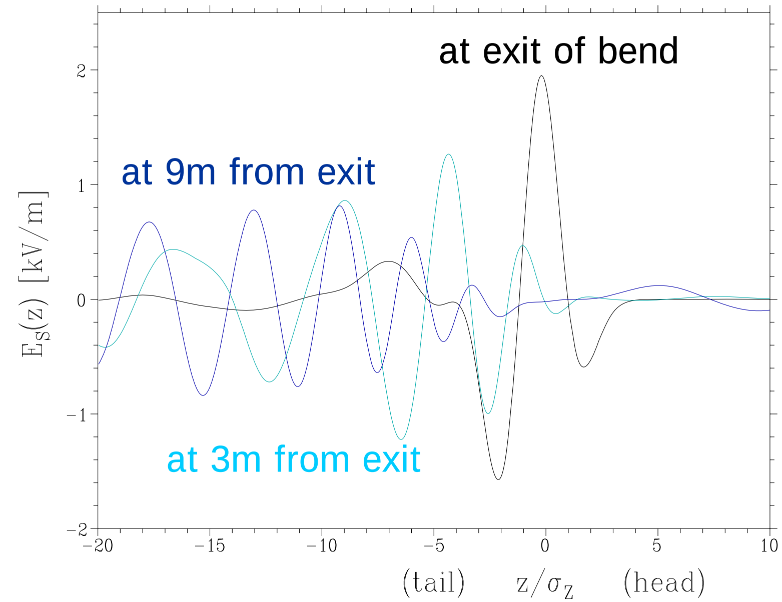
Low frequency limit ⇒ Resistive wall impedance : $Z_{||}(k) = \frac{\sqrt{Z_0/\sigma_c}}{2\pi r} e^{-i\pi/4}\sqrt{k}$

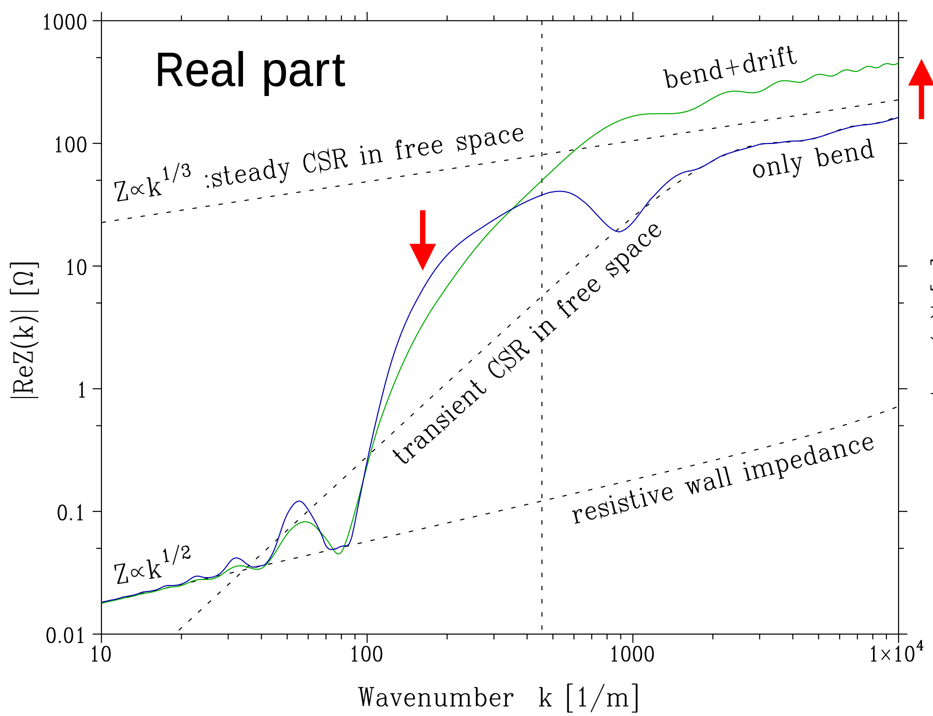
High frequency limit ⇒ Steady CSR in free space : $Z_{||}(k) = \frac{Z_0}{2\pi}\left(\frac{k}{3\rho^2}\right)^{1/3}\Gamma\left(\frac{2}{3}\right) e^{i\pi/6}$

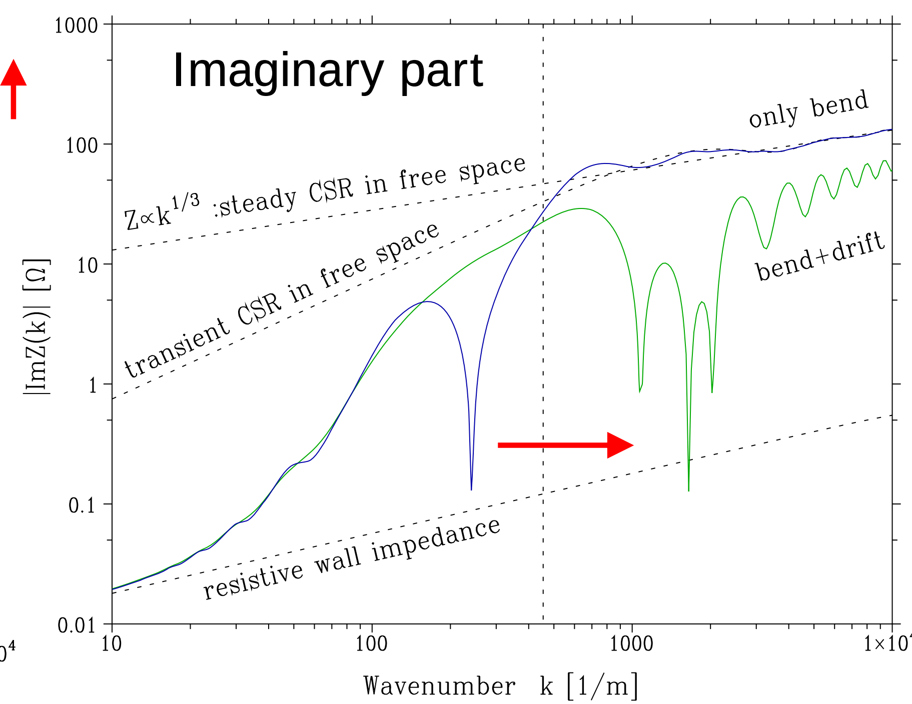
CSR goes out a bend and propagates in the drift space, where particles are still affected with CSR.



1. Longitudinal delay because of reflection
2. Sinusoidal behavior as it propagates





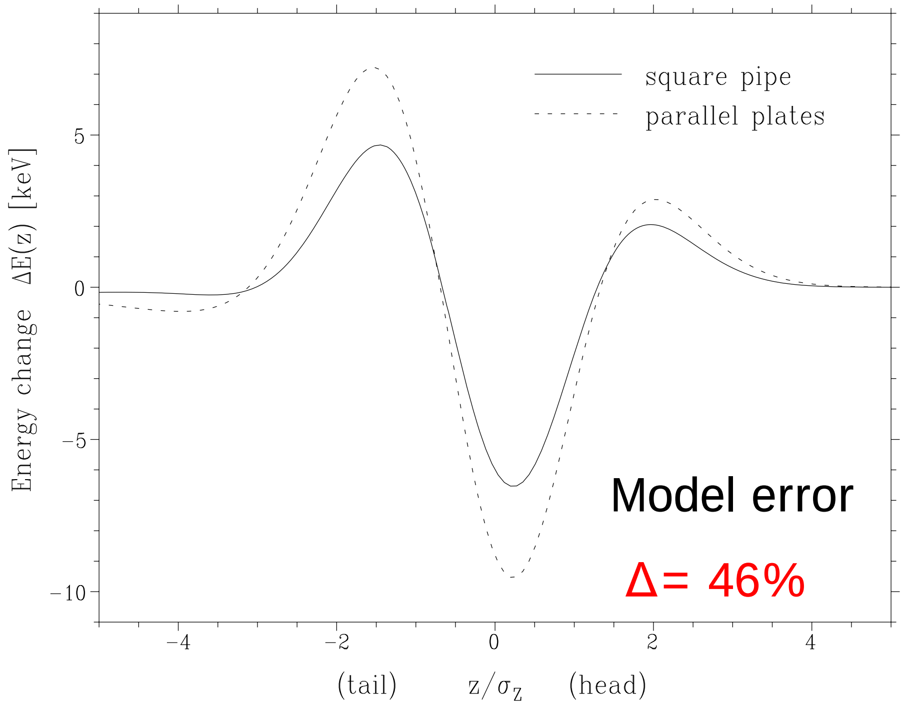


Error of parallel plates model

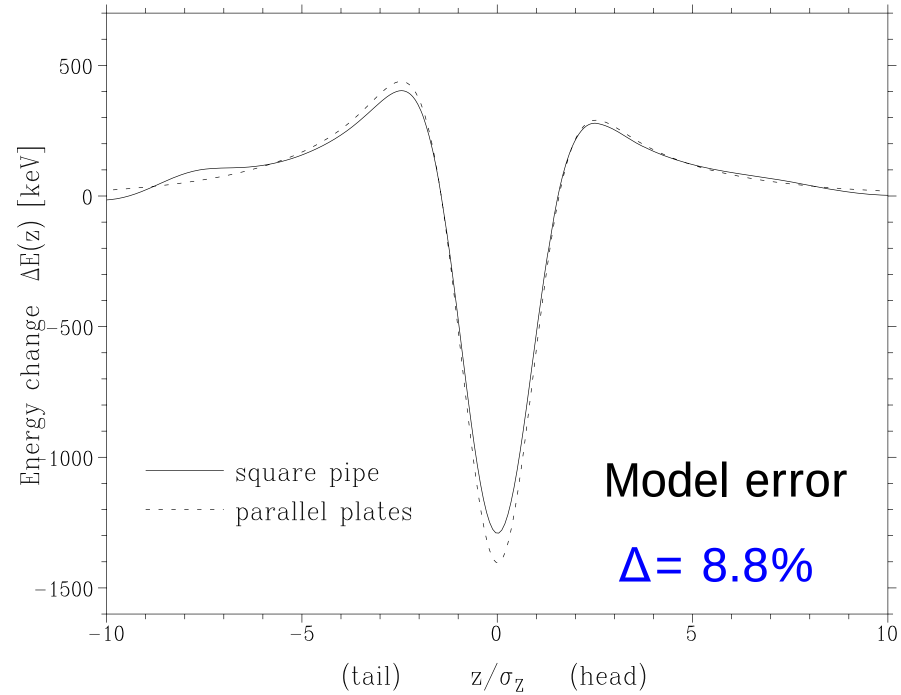
Chamber size (full width × full height)

- Square pipe : 94 ×94 mm^2 (solid line)
- Parallel plates : 400 ×94 mm^2 (dashed line)

Bunch length : $\sigma_z = 3\text{mm}$



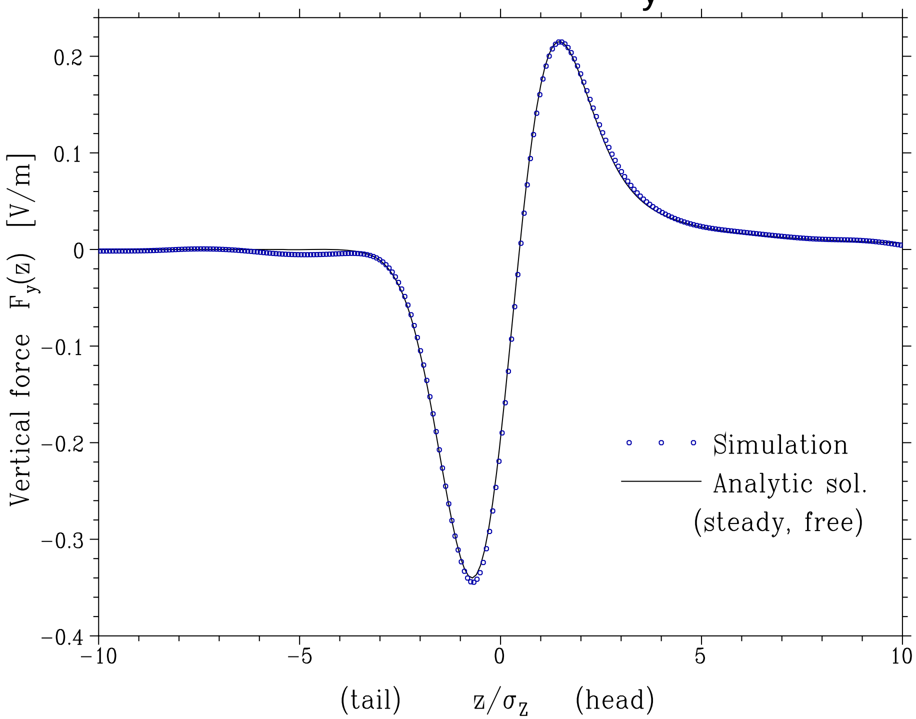
$\sigma_z = 0.3\text{mm}$



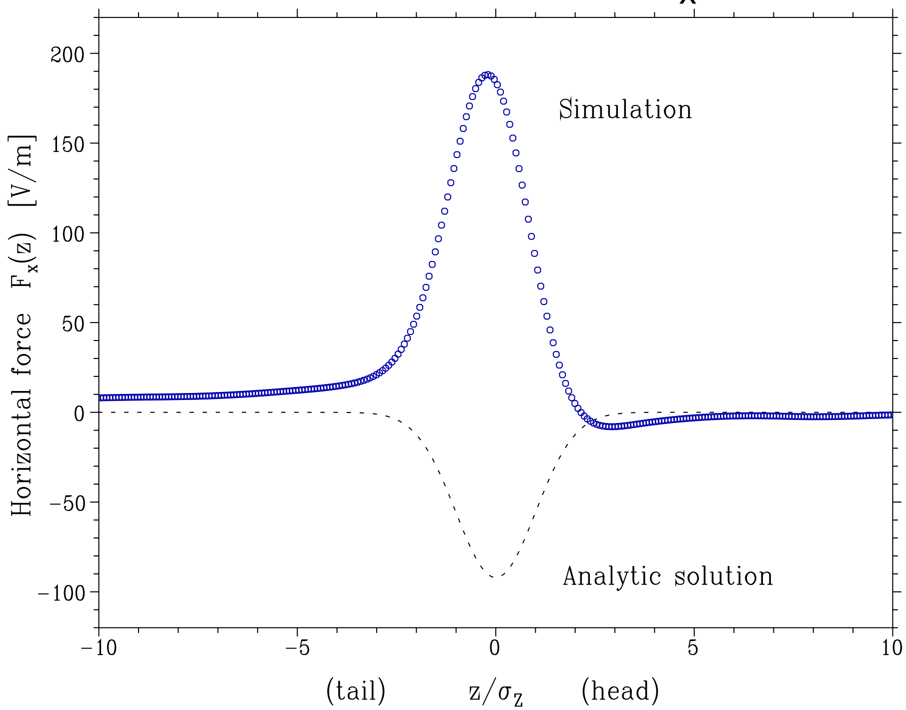
Parallel plates model may work for very short bunch but we should consider a beam pipe for storage rings.

Transverse force of CSR

Vertical force F_y



Horizontal force F_x



$$F_y(z) = -y\frac{Ne^2}{4\pi\varepsilon_0(3\rho^2)^{2/3}}\frac{\partial}{\partial z}\int_{-\infty}^{z} dz' \frac{\lambda(z')}{(z-z')^{2/3}}$$

$$F_x = -\frac{\partial V_0}{\partial x} - e\frac{dA_x}{ds} + \frac{eA_s}{x+\rho}$$

neglected

$$F_x(z) = -\frac{Ne^2}{2\pi\varepsilon_0\rho}\lambda(z) - 3x\frac{Ne^2}{4\pi\varepsilon_0(3\rho^2)^{2/3}}\frac{\partial}{\partial z}\int_{-\infty}^{z} dz' \frac{\lambda(z')}{(z-z')^{2/3}}$$

Ya.S.Derbenev, V.D.Shiltsev, SLAC-PUB-7181 (1996)

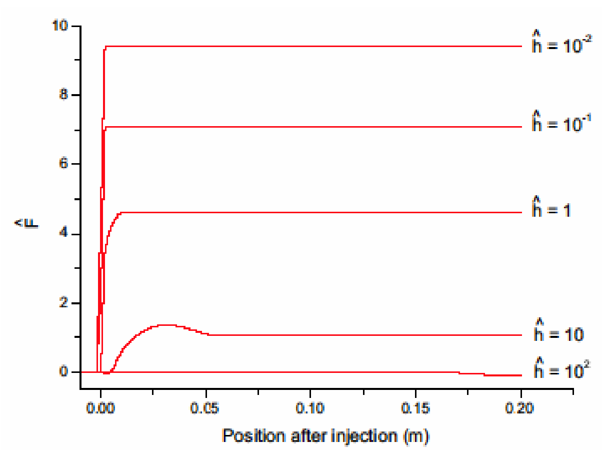
“Transverse effects of Microbunch Radiative Interaction”

Horizontal force on a curved trajectory

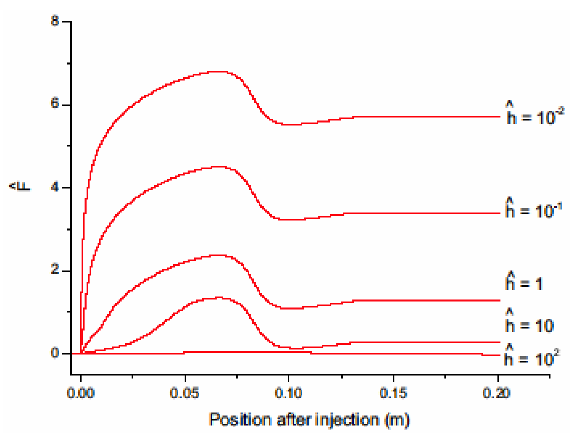
Horizontal force consists of not only forward waves but also backward waves.

G.Geloni, E.Saldin, E.Schneidmiller, M.Yurkov, DESY 03-165 (2003)

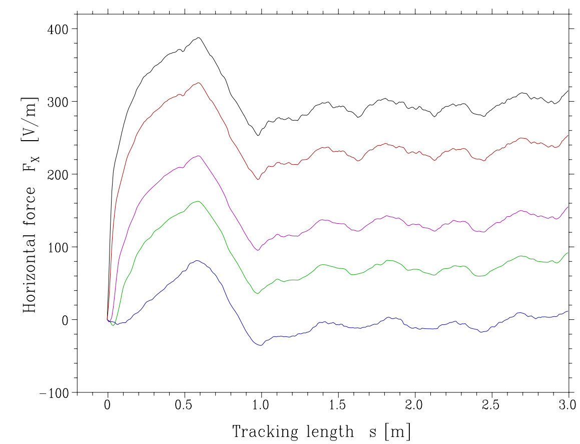
forward



backward



my result



Since CSR is emitted forward, the backward component in Fx is not the radiation but a kind of space charge force.

"centrifugal space charge force", "Talman force"

We neglect backward waves in the paraxial approximation, the horizontal force may be incorrect in our approach.

KEKB factory (e$^+$e$^-$ storage ring collider)

	KEKB LER	SuperKEKB LER
Bunch length	6 mm	3 mm
Bunch current (charge)	1.4 mA (~14 nC)	1.9 mA (~19 nC)

Upgrade plan to SuperKEKB (2009)

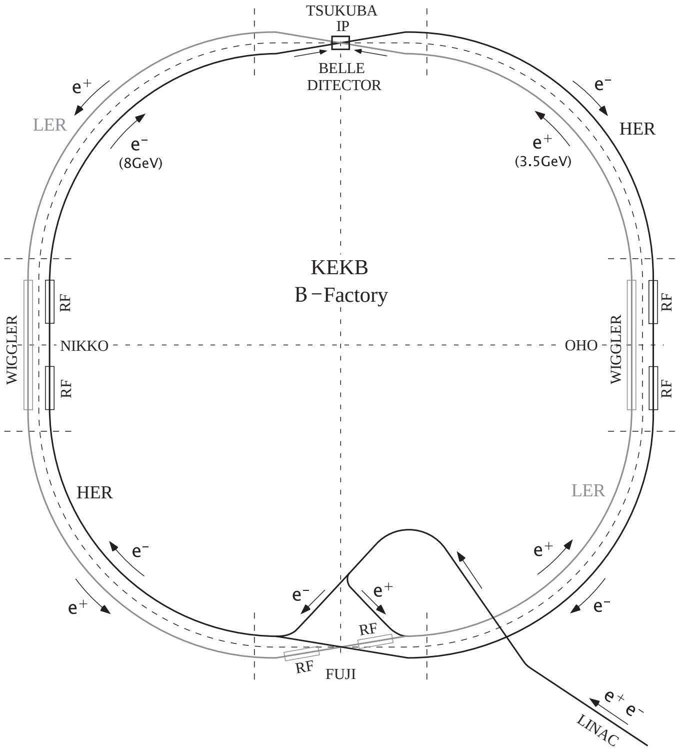
$L=4 \times 10^{35}$

We will keep using present magnets to save money and R&D time.

Bending radius

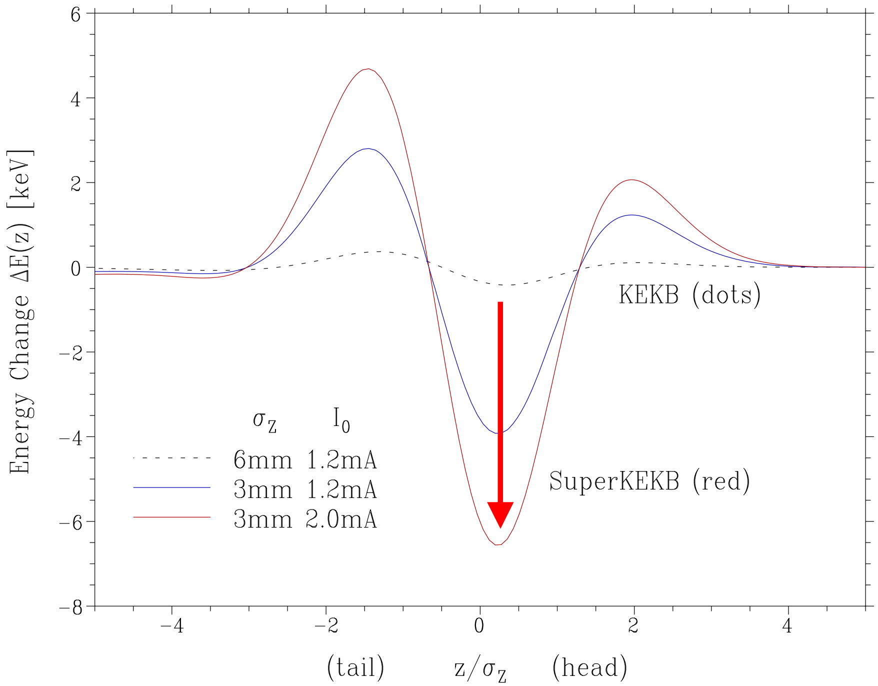
LER (positron): R=16.31m
HER (electron): R=104.5m

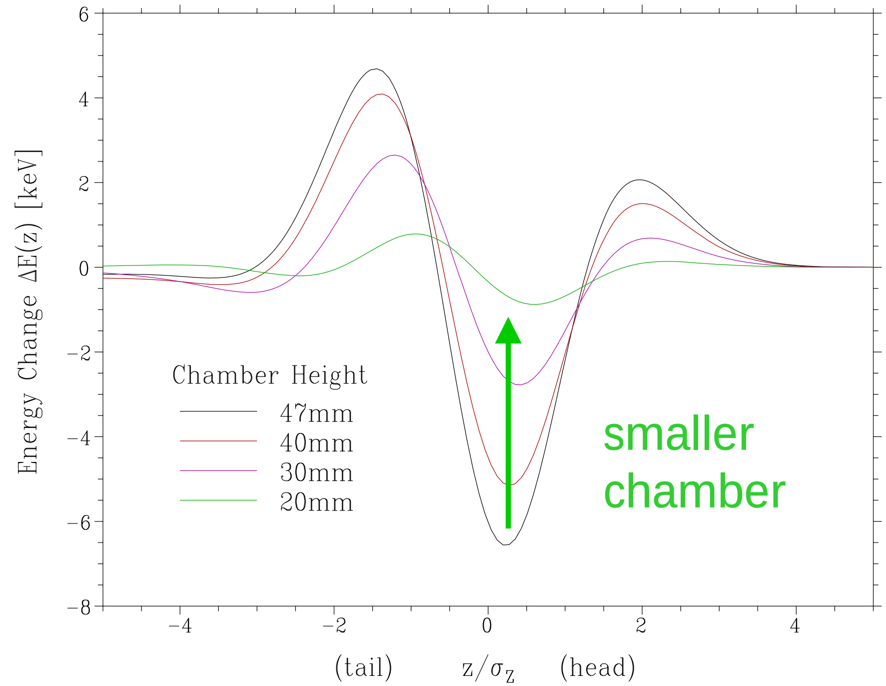
Positron bunch will be affected with CSR.



CSR in SuperKEKB

Energy change due to CSR (Longitudinal wakefield for a single bend)





KEKB → SuperKEKB

CSR effect is 14 times larger

Small chambers suppress CSR.

We will make new vacuum chamber to suppress electron cloud effect.

Variation of bunch profile

In a storage ring, bunch distribution changes by wakefield and damping.

CSR depends on the longitudinal bunch shape, we must consider the variation of bunch shape

Initial distribution (macro-particles)

Green function of CSR (thin Gaussian distribution)

↓ ← Initial distribution (macro-particles histogram)

Field calculation (superposition)

↓

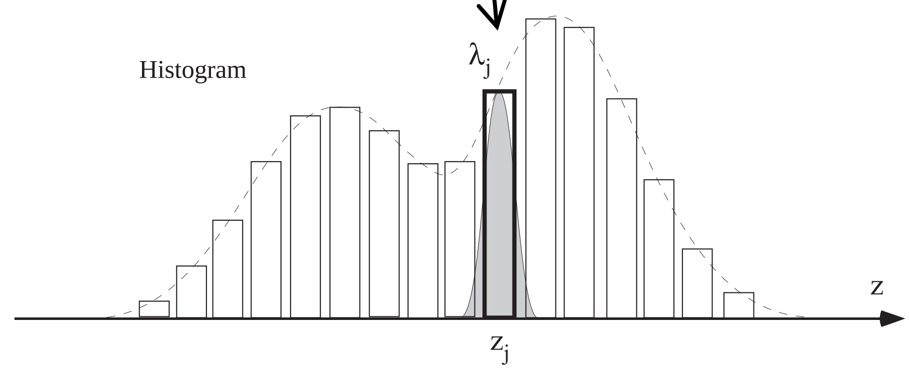
Equation of motion

↓

New charge distribution

iteration

Macro-particle tracking



bins replaced by Green function

Microwave instability due to CSR

Equations of longitudinal motion

resistive pipe considered

$$\begin{cases} z' = -\eta\delta \\ \delta' = \frac{(2\pi\nu_s)^2}{\eta C^2} z - \frac{2U_0}{CE_0}\delta + \text{Quantum Excitation} + \text{CSR} + (\text{RW}) \end{cases}$$

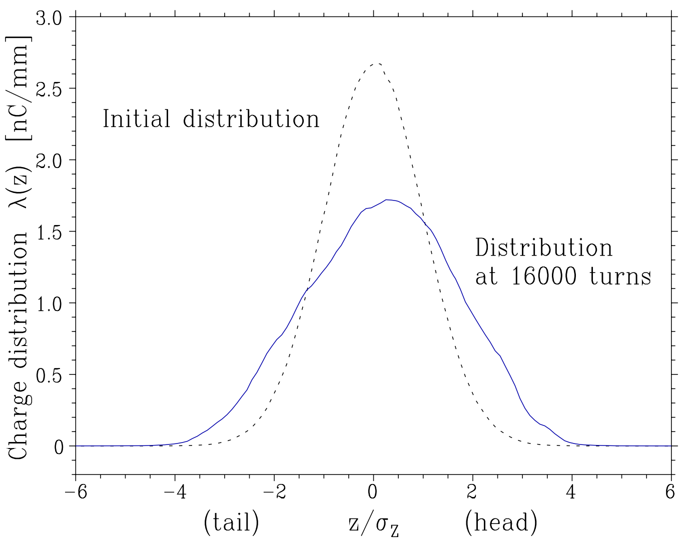
Resistive wall wake in the drift space

- 134 arc bends are considered for CSR.
- Wiggler is neglected (should be considered).
- Wiggler is taken into account in rad. damping.
- Copper pipe of square cross section
  (Actual chamber is a round pipe)
- Option: Resistive wall wake in the drift space

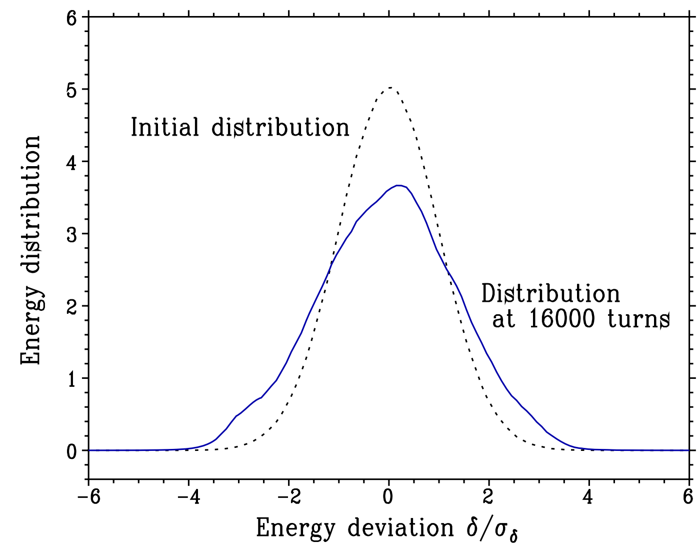
- parameters
  $E_0 = 3.5\,\text{GeV}$
  $C = 3016.26\,\text{m}$
  $\sigma_z = 3\,\text{mm}$
  $\sigma_\delta = 7.1\times10^{-4}$
  $V_{\text{rf}} = 15\,\text{MV}$
  $\omega_{\text{rf}} = 508.887\,\text{Hz}$
  $h = 5120$
  $\alpha = 2.7\times10^{-4}$
  $U_0 = 1.23\,\text{MeV/turn}$
  $\nu_s = 0.031$

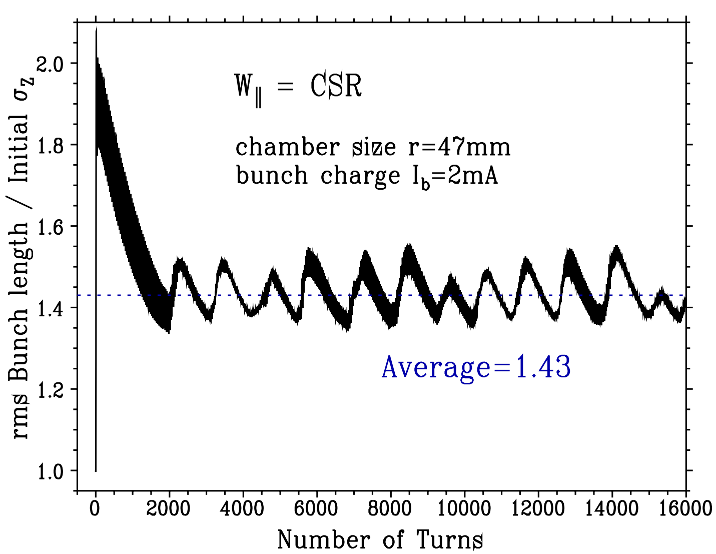
Chamber half size: r=47mm (only CSR)

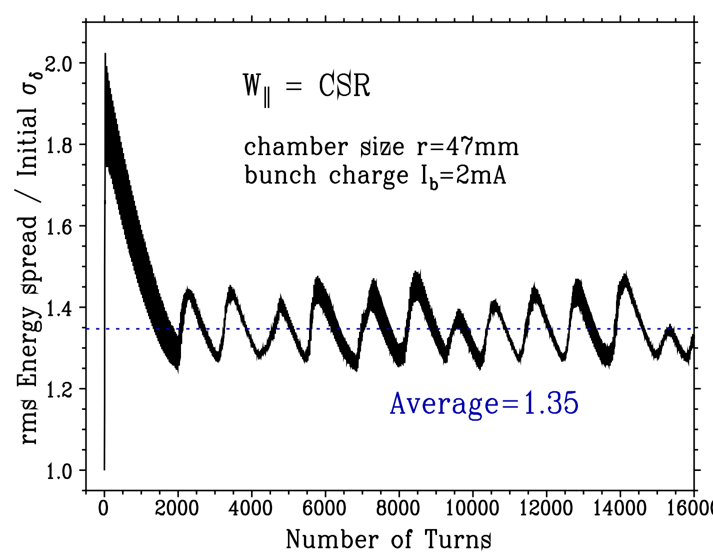
Charge distribution



Energy distribution



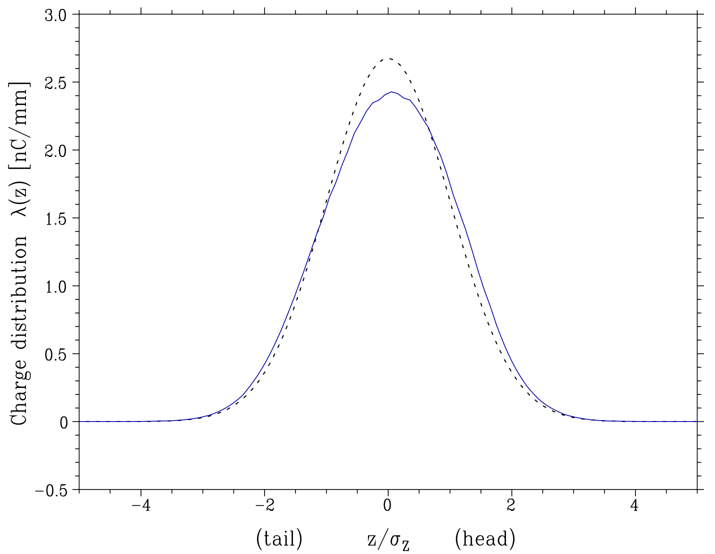




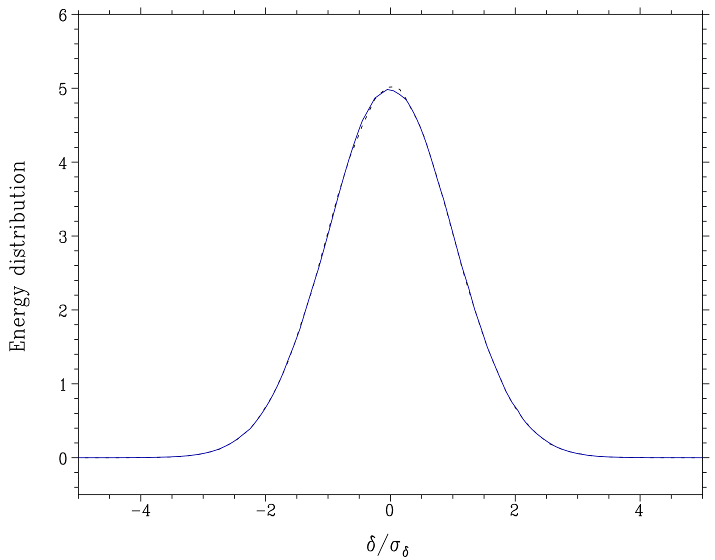
Chamber half size: r=25mm

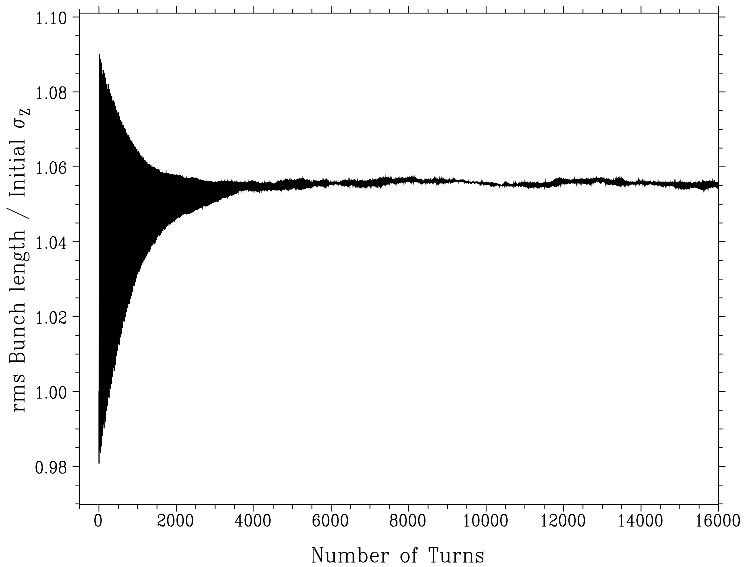
(only CSR)

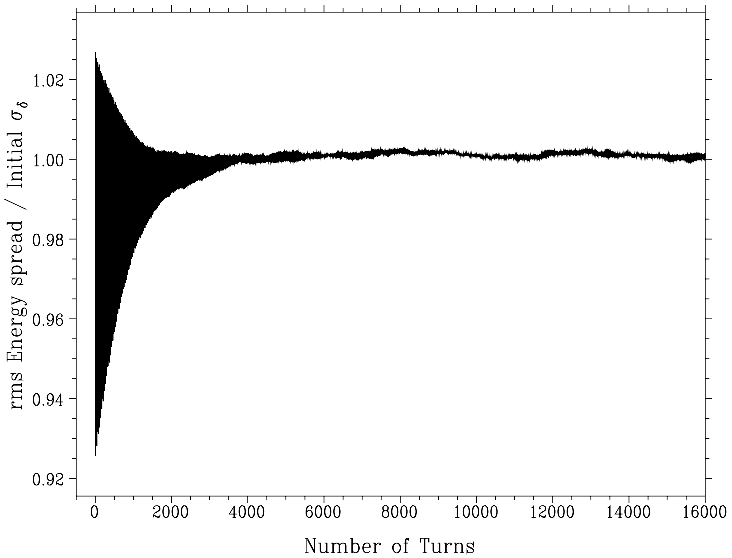
Charge distribution



Energy distribution





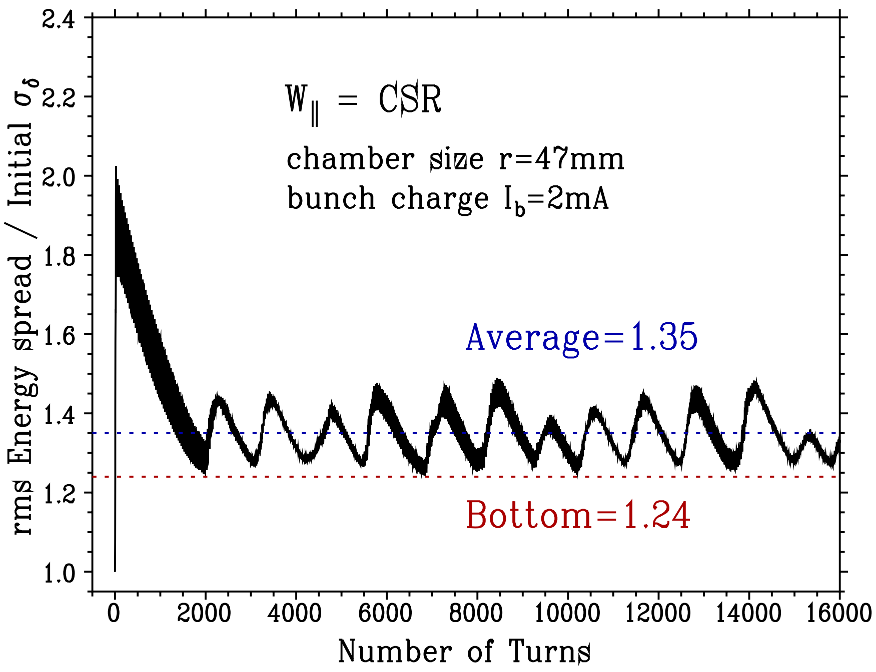


Saw-tooth instability

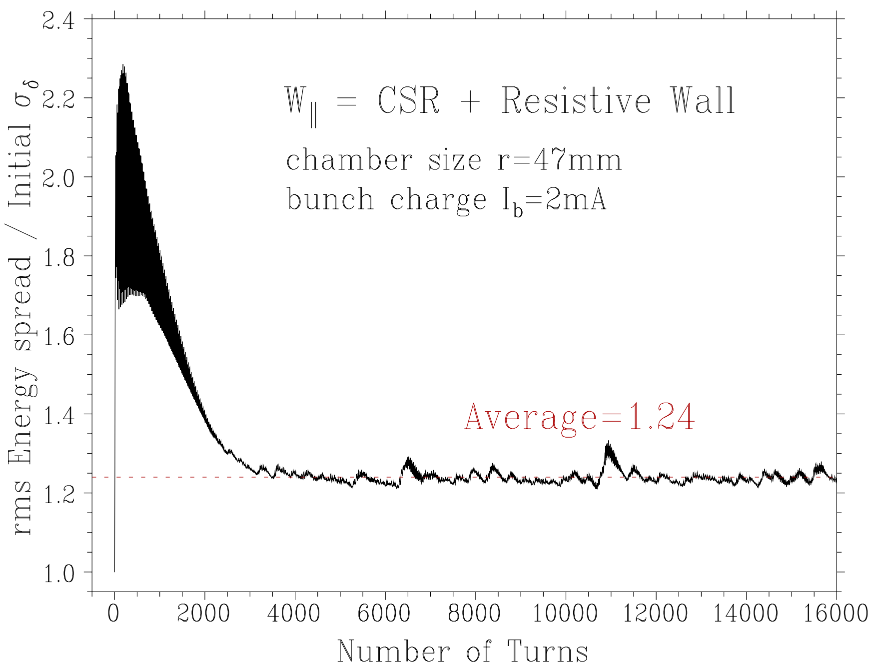
Resistive wall wakefield reduces the saw-tooth amplitude.

rms energy spread vs number of turns

only CSR

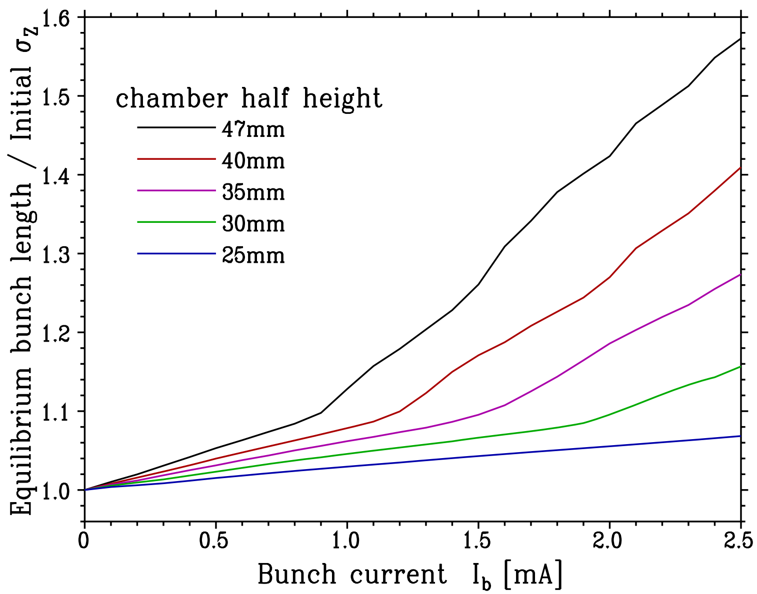


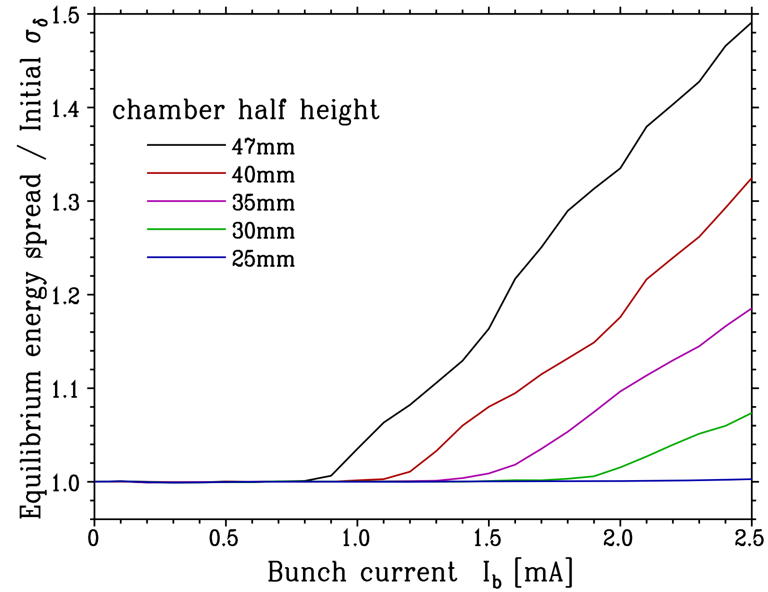
CSR + RW wake in the drift space



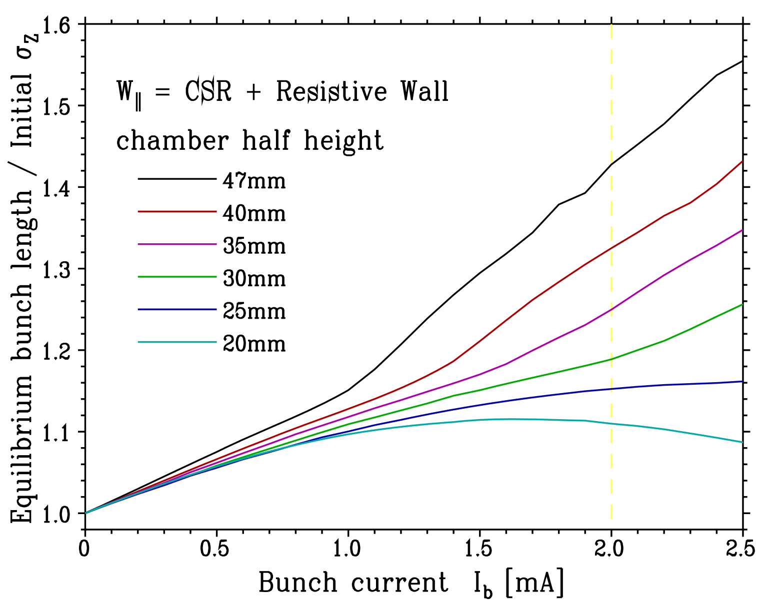
Bunch length, Energy spread vs bunch charge

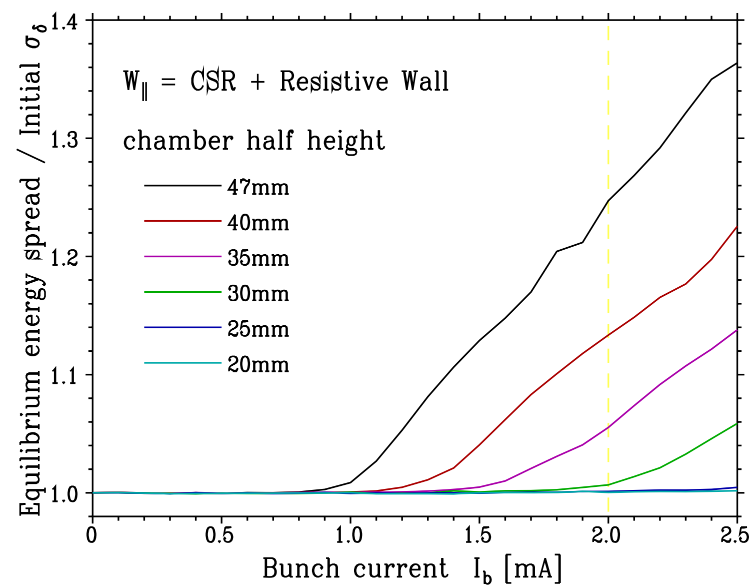
only CSR



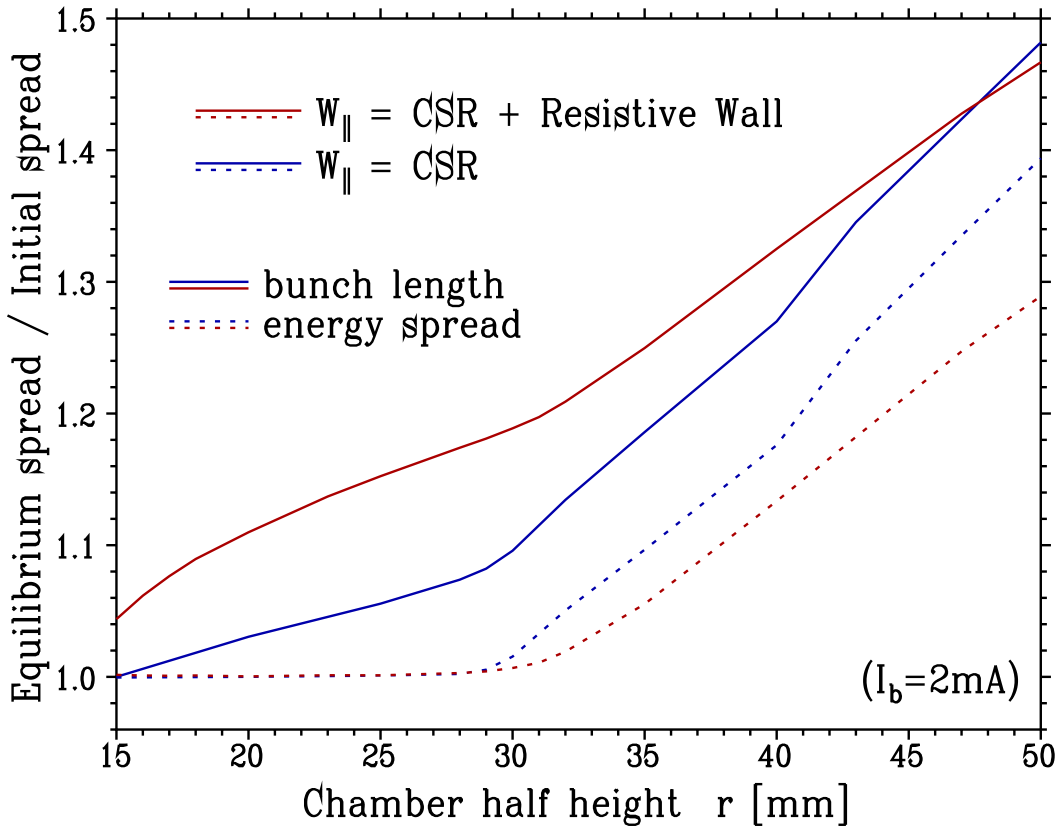


CSR + RW in drift space





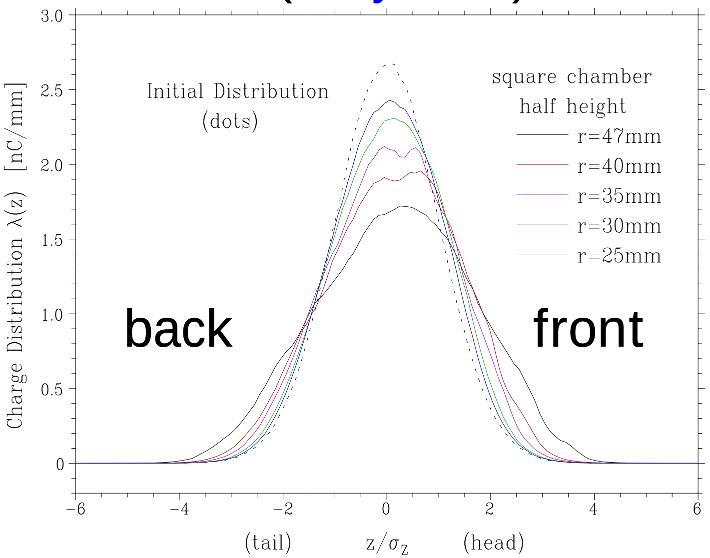
Vacuum chamber size



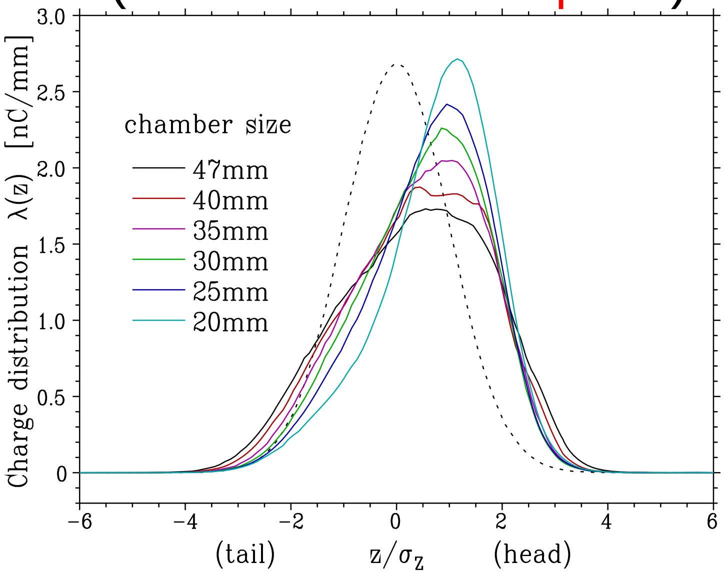
Resistive wall wakefield does not change the instability threshold.

Bunch profile

(only CSR)

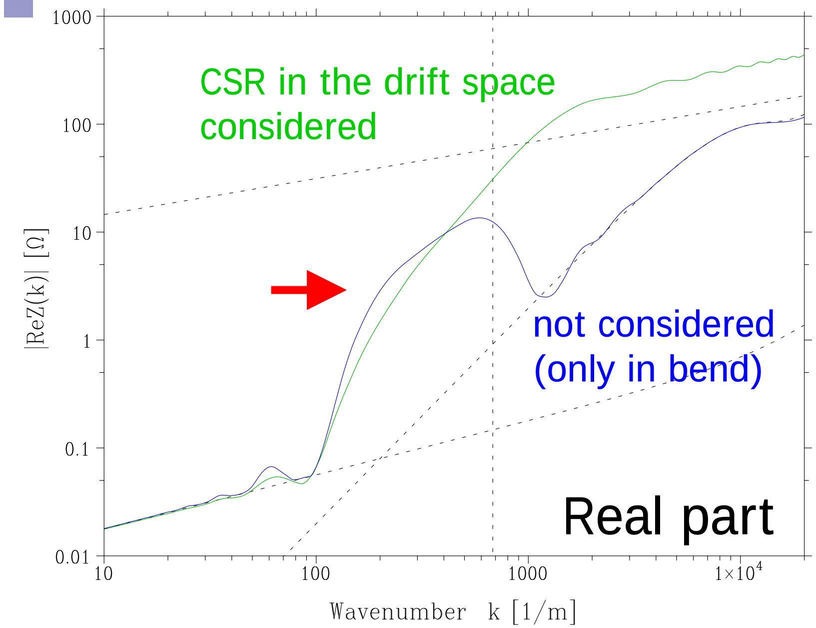


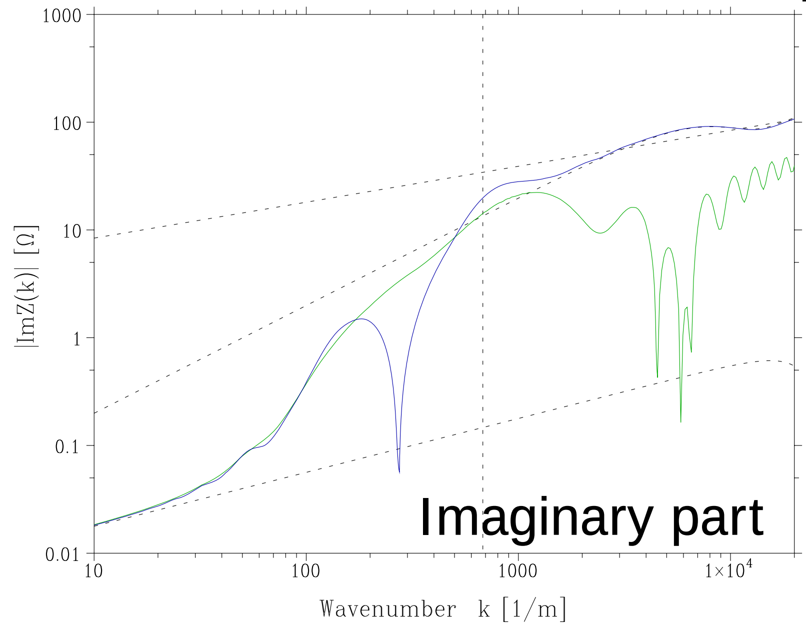
(CSR + RW in drift space)

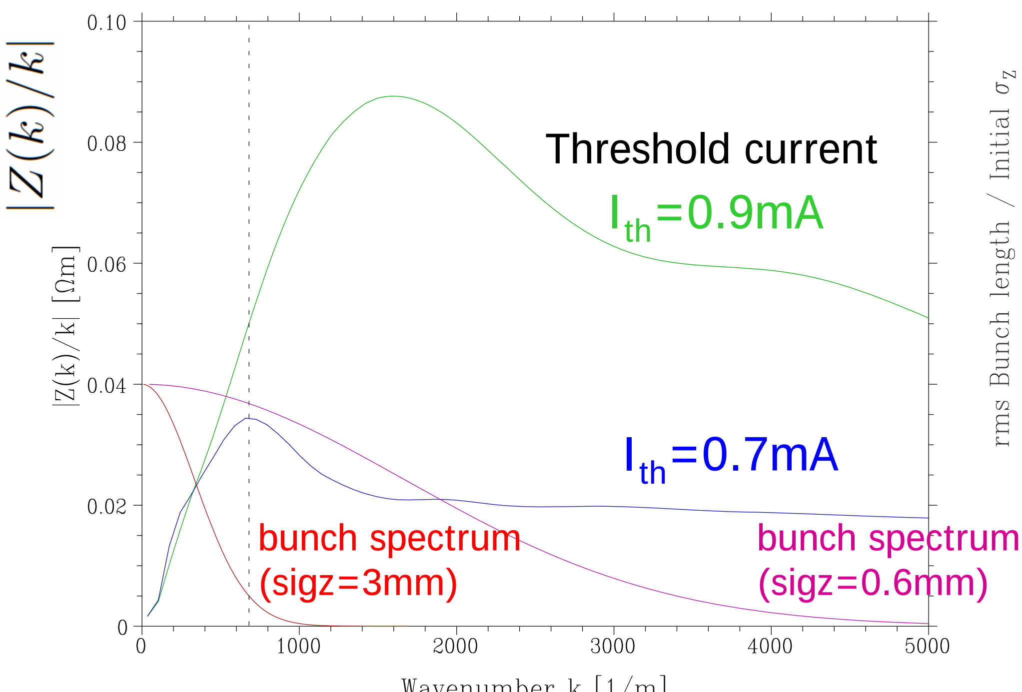


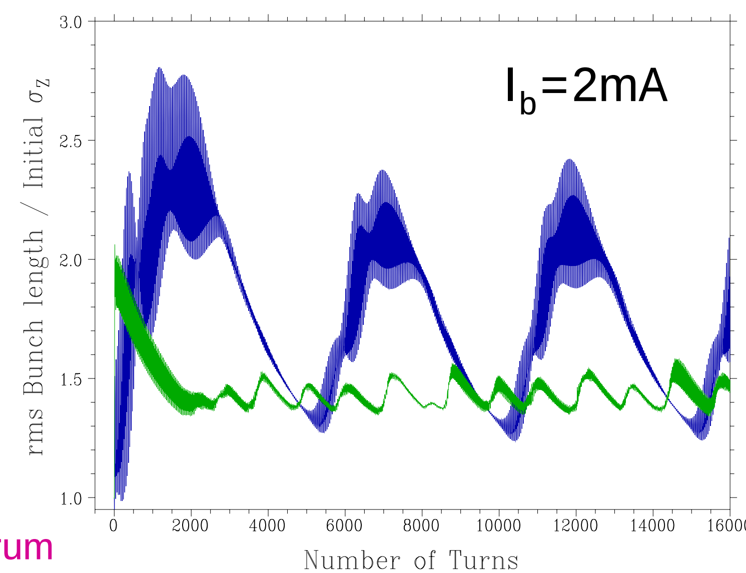
Bunch leans forward because of energy loss due to the resistive wall wakefield.

CSR in the drift space



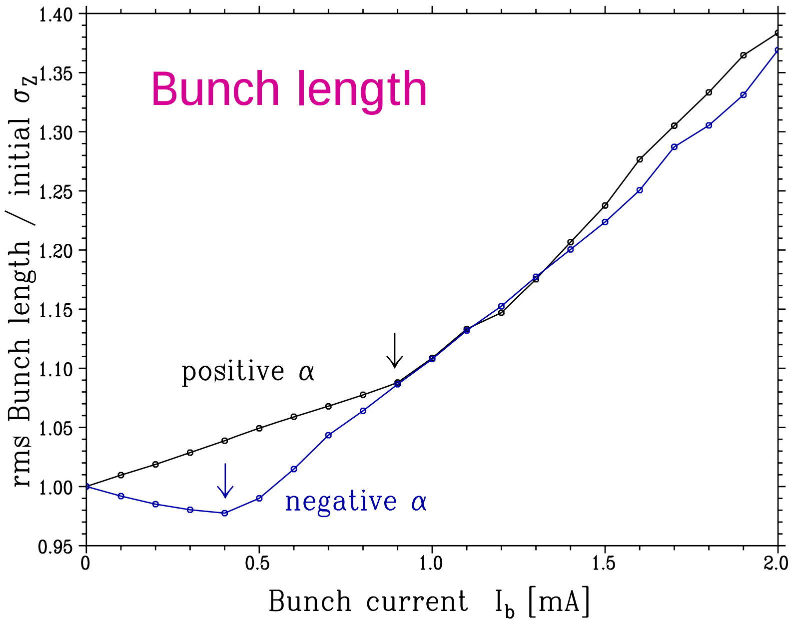


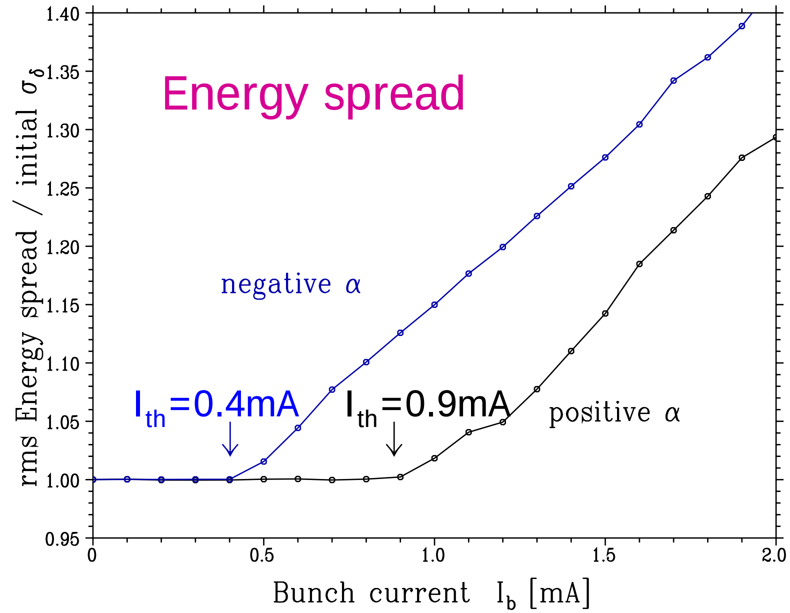




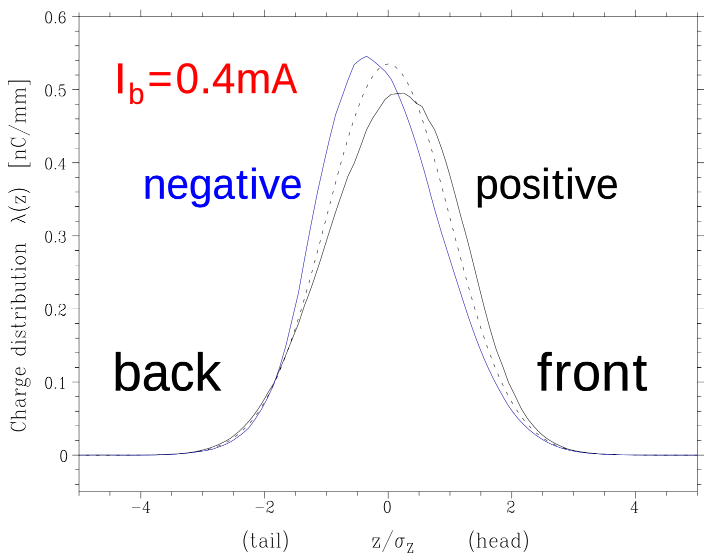
CSR in the drift space relaxes the longitudinal instability.

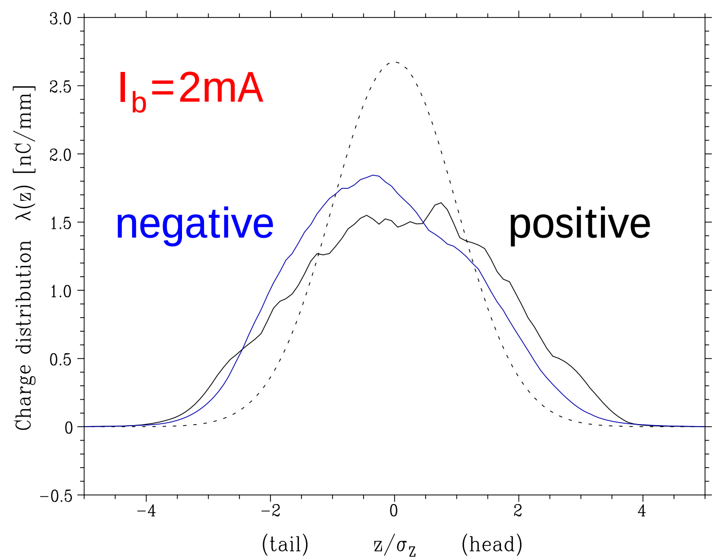
Negative momentum compaction (only CSR considered, no RW wake in drift space)





Bunch profile

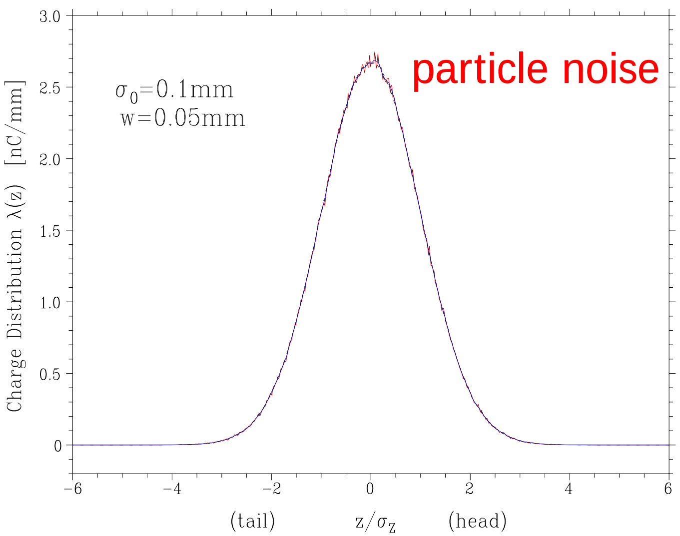


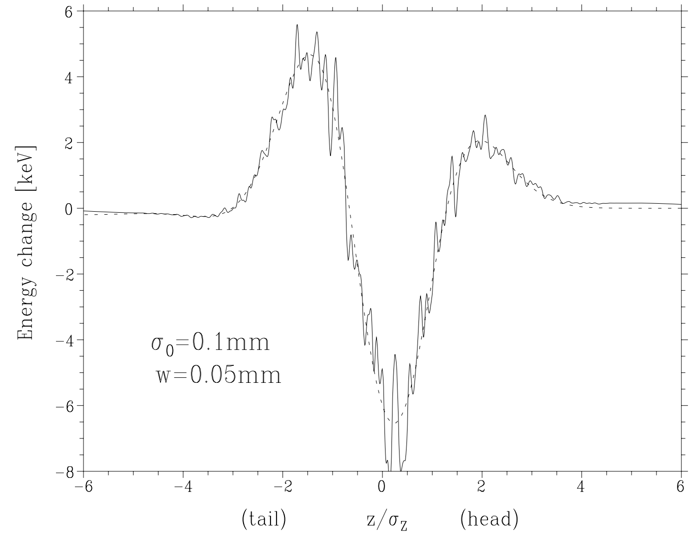


Numerical problem

Bunch profile

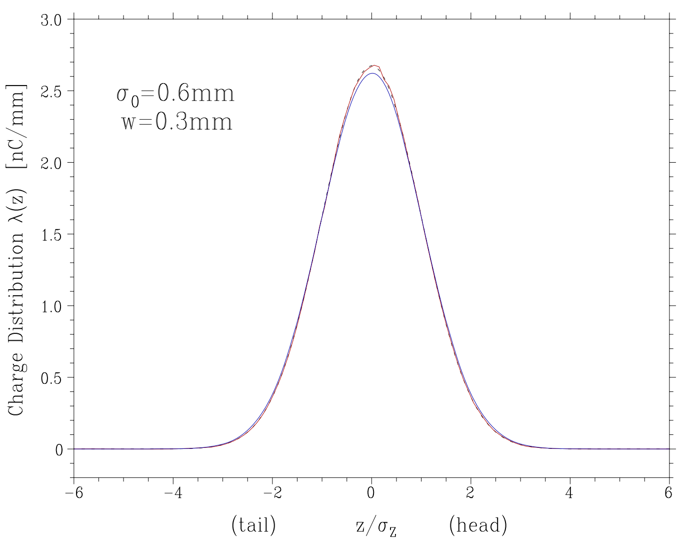
Longitudinal CSR wake

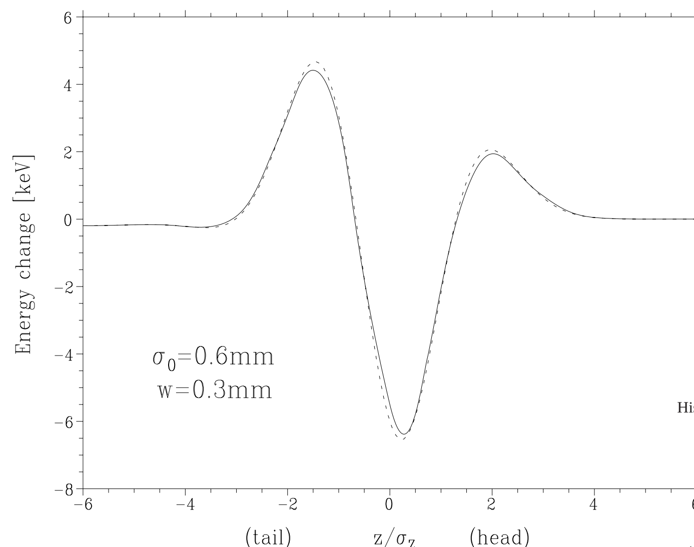




$\sigma_0 = 0.1\text{mm}$

Width of Gaussian Green function

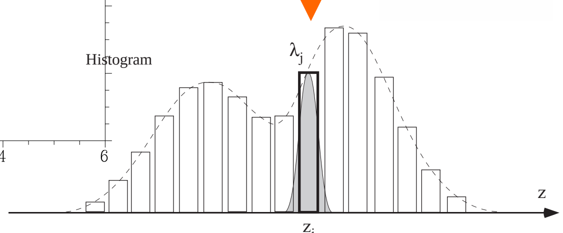




$\sigma_0 = 0.6\text{mm}$

$\sigma_z = 3\text{mm}$

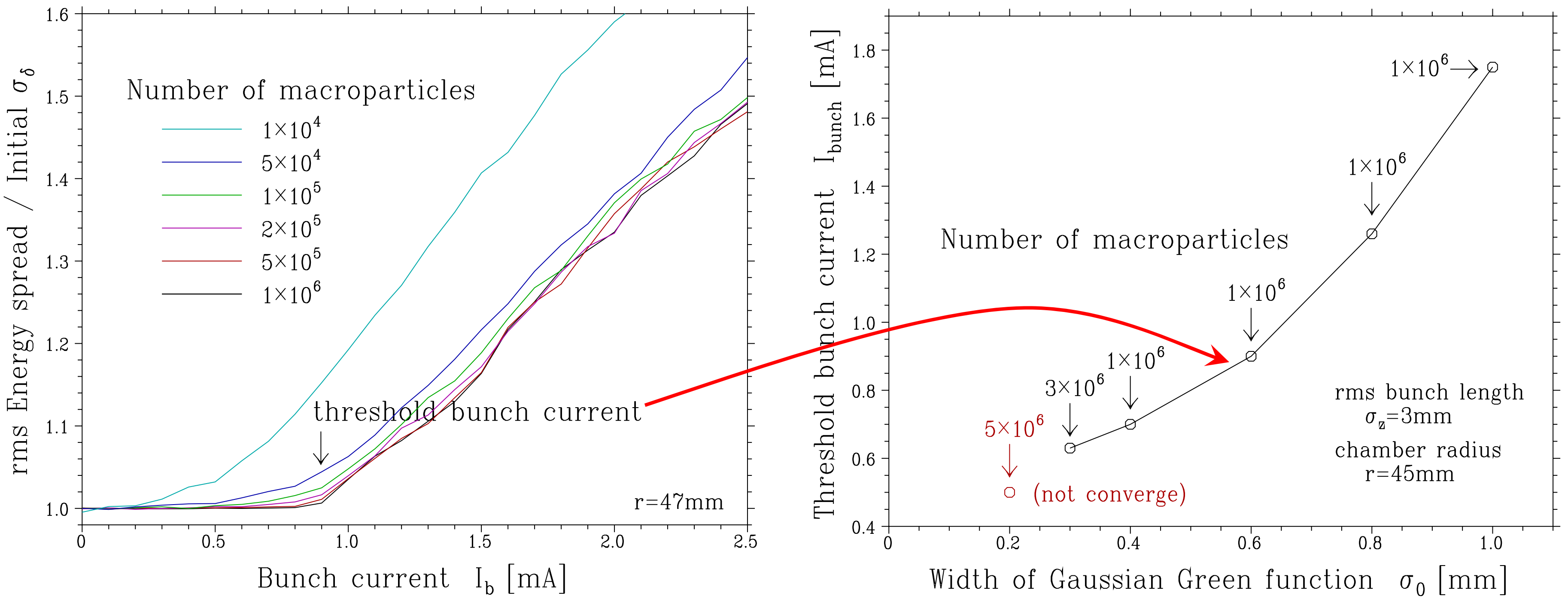
$N = 10^6$



Instability threshold for Green function width

Energy spread vs bunch current

Threshold vs Green function width



Width of Green function

$\sigma_0 = 0.6\text{mm} \; (= \sigma_z/5)$

Threshold current does not converge for Green function width.

We cannot distinguish between instability and particle noise.

Conclusions

- CSR calculation is performed by paraxial approximation.
  - Shielding by a beam pipe, Transient state, Resistive wall
  - CSR in the drift space
- Our approach has a defect in the horizontal space charge force.
  - backward wave ignored
- CSR will induce longitudinal instability in SuperKEKB positron ring.
  - The threshold bunch current is less than 0.9mA in the present chamber (r=47mm).
  - Vacuum chamber of r=28mm will suppress the CSR effect. However, the small chamber may cause side effects.
  - CSR in the drift space relaxes the longitudinal instability.
  - Particle tracking does not work for microwave instability, threshold current is not clear.