Conditioning and symplecticity

Assume that the conditioner does not introduce coupling between the vertical and horizontal planes, and consider only the horizontal plane with the initial values of coordinates (x_0, x'_0) at the entrance, and the final values (x, x') at the exit.

Instead of using variables x_0, x'_0 and x, x', introduce new variables u_0, v_0, and u, v

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = Q_0 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = Q \begin{pmatrix} x \\ x' \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix},$$

with β_0, α_0 and β, α the Twiss parameters.

The map from u_0, v_0, z_0, δ_0 to u, v, z, δ is symplectic. In linear approximation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} ,$$

where

$$A = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} ,$$

with ψ the betatron phase advance.

“One-Phase” Conditioner

Contribution of the x-coordinate $x_0^2/(\beta\epsilon_0)$ to the parameter r^2 is equal to u_0^2/ϵ_0,

$$r^2 \longrightarrow \frac{u_0^2}{\epsilon_0}\,,$$

and the conditioning requirement can be written as

$$\delta = \delta_0 + \frac{1}{2} b u_0^2\,,$$

where $b = a/\sigma_{z0}$.

Symplecticity and generating function

Symplecticity means that u_0, v_0, z_0, δ_0 and u, v, z, δ are related via a canonical transformation.

We use a generating function which depends on old coordinates u_0 and z_0 and new momenta v and δ, $F(u_0, z_0, v, \delta)$.

$$v_0 = \frac{\partial F}{\partial u_0}\,, \qquad \delta_0 = \frac{\partial F}{\partial z_0}\,, \qquad u = \frac{\partial F}{\partial v}\,, \qquad z = \frac{\partial F}{\partial \delta}\,.$$

In paraxial approximation all coordinates and momenta are considered small and we can expand F in Taylor series.

$$F \approx F_2 + F_3 + \dots\,,$$

where F_2 is a quadratic, and F_3 is a cubic function of the coordinates and momenta.

The function F_2 should generate our linear map for u and v with a unit transformation for z and δ

$$F_2 = \frac{1}{2}(u_0^2 + v^2)\tan\psi + u_0 v \sec\psi + \delta z_0 \,.$$

The function F_3 involves 2^{nd}-order abberations in the system. We chose only the term responsible for the conditioning:

$$F_3 = -\frac{1}{2} b z_0 u_0^2 \,.$$

We find

$$\delta_0 = \delta - \frac{1}{2} b u_0^2 \,.$$

We also have

$$
\begin{aligned}
z &= z_0 \\
v_0 &= u_0 \tan\psi + v \sec\psi - b z_0 u_0 \,, \\
u &= v \tan\psi + u_0 \sec\psi \,.
\end{aligned}
$$

These equations can be easily solved for u and v:

$$
\begin{aligned}
u &= u_0 \cos\psi + v_0 \sin\psi + b z_0 u_0 \sin\psi \,, \\
v &= -u_0 \sin\psi + v_0 \cos\psi + b z_0 u_0 \cos\psi \,.
\end{aligned}
$$

For the single phase solenoid conditioner $\psi = 2n\pi$, $\beta_0 = \beta$, $\alpha_0 = \alpha = 0$, and this equation agrees with Paul's equations for "one-phase" conditioner (with $R_{56} = 0$).

Calculate the projected emittance increase of the beam due to the conditioning:

$$\epsilon_x^2 = \langle u^2 \rangle \langle v^2 \rangle - \langle uv \rangle^2$$

where the averaging is

$$\langle \ldots \rangle = \int \frac{du_0\, dv_0}{2\pi\epsilon_{x0}} e^{-(u_0^2+v_0^2)/2\epsilon_{x0}} \frac{dz_0}{\sqrt{2\pi}\sigma_{z0}} e^{-z_0^2/2\sigma_{z0}^2} \ldots$$

Result

$$\epsilon_x^2 = \epsilon_{x0}^2 (1 + b^2 \sigma_{z0}^2) = \epsilon_{x0}^2 (1 + a^2) \,.$$

For large a

$$\frac{\epsilon_x}{\epsilon_{x0}} \approx a$$

Conclusions

- Due the symplecticity of the map, an FEL conditioner unavoidably generates differential focusing along the bunch which results in the emittance growth that is directly related to the conditioning parameter a. Any attempt to correct this abberation downstream would result in ruining the conditioning. We demonstrated this on a solenoid conditioner, and proved for a general symplectic one-phase conditioner.

- The parameter a is large for modern short-wavelength FELs and makes the emittance growth unacceptable due to the conditioning.

- Simulations show that for two-phase conditioner, the effect of the emittance growth is even worth than for one-phase conditioner.